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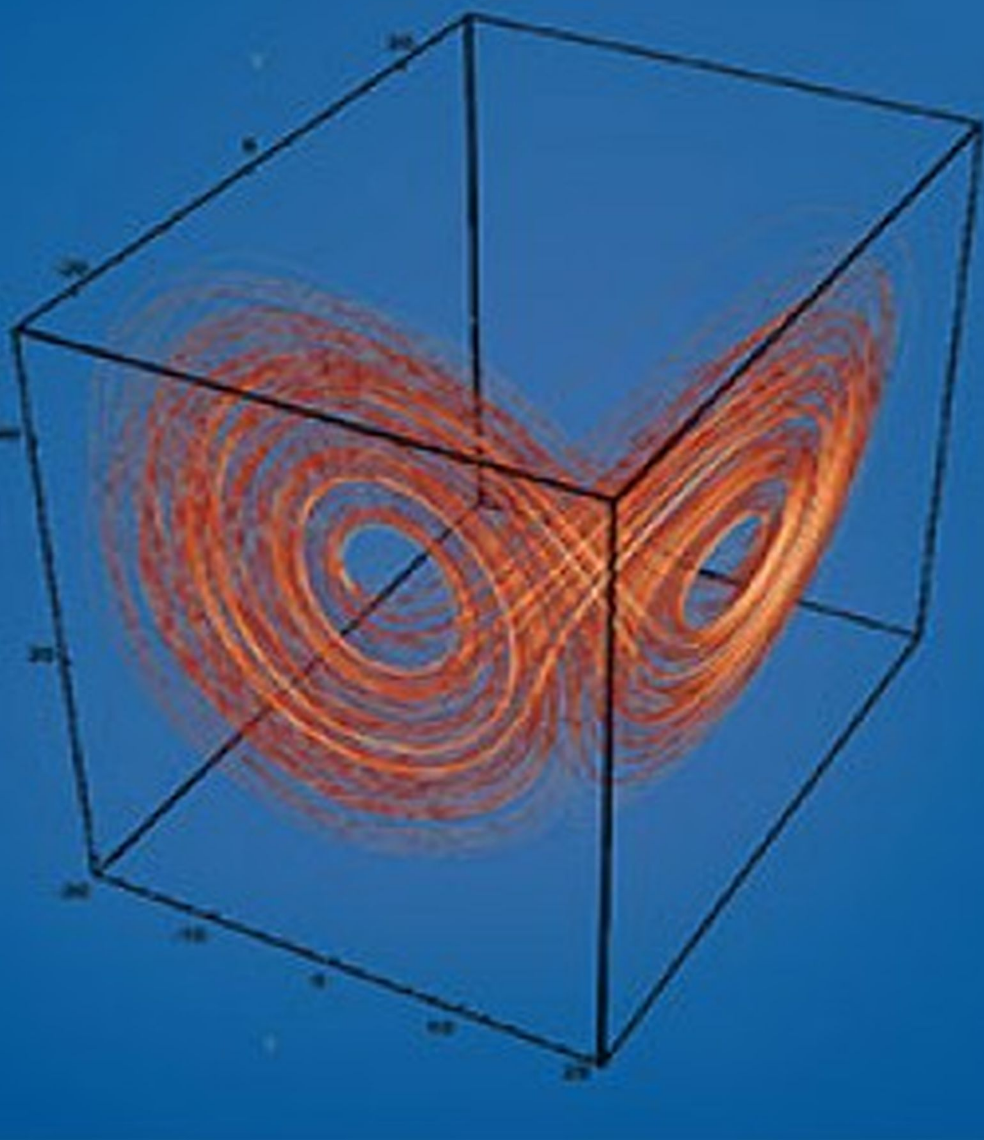
Mastering Differential Equations: The Visual Method

Professor Robert L. Devaney
Boston University



Mastering Differential Equations: The Visual Method

Robert L. Devaney, Ph.D.

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PUBLISHED BY:

THE GREAT COURSES
Corporate Headquarters
4840 Westfields Boulevard, Suite 500
Chantilly, Virginia 20151-2299
Phone: 1-800-832-2412
Fax: 703-378-3819
www.thegreatcourses.com

Printed in the United States of America

Robert L. Devaney, Ph.D.

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Professor Robert L. Devaney is Professor of Mathematics at Boston University. He received his undergraduate degree from the College of the Holy Cross in 1969 and his Ph.D. from the University of California, Berkeley, in 1973. He taught at Northwestern University and Tufts University before moving to Boston University in 1980.

Professor Devaney's main area of research is dynamical systems, primarily complex analytic dynamics, but also more general ideas about chaos in dynamical systems. Lately, he has become intrigued with the incredibly rich topological aspects of dynamics, including such things as indecomposable continua, Sierpinski curves, and Cantor bouquets.

The author of more than 100 research papers and a dozen pedagogical papers in the field of dynamical systems, Professor Devaney is also the coauthor or editor of 13 books. These include *An Introduction to Chaotic Dynamical Systems*, a text for advanced undergraduate and graduate students in mathematics and researchers in other fields; *A First Course in Chaotic Dynamical Systems*, written for undergraduate college students who have taken calculus; and the series of 4 books collectively called *A Tool Kit of Dynamics Activities*, aimed at high school students and teachers as well as college faculty teaching introductory dynamics courses to science and nonscience majors. For the last 18 years, Professor Devaney has been the principal organizer and speaker at the Boston University Math Field Days. These events bring more than 1000 high school students and their teachers from all around New England to the campus for a day of activities aimed at acquainting them with what's new and exciting in mathematics.

Professor Devaney has delivered more than 1400 invited lectures on dynamical systems and related topics in every state in the United States and in more than 30 countries on 6 continents. He has designed lectures for research-

level audiences, undergraduate students and faculty, high school students and faculty, and the general public. He has also been the chaos consultant for several theaters' presentations of Tom Stoppard's play *Arcadia*. In 2007 he was the mathematical consultant for the movie *21*, starring Kevin Spacey.

Since 1989 Professor Devaney has been director of the National Science Foundation's Dynamical Systems and Technology Project. The goal of this project is to show students and teachers how ideas from modern mathematics such as chaos, fractals, and dynamics, together with modern technology, can be used effectively in the high school and college curriculum. As part of this project, Professor Devaney and his students and colleagues have developed numerous computer programs for exploring dynamical systems. Another long-standing National Science Foundation project has been an attempt to revitalize the study of ordinary differential equations by thoroughly incorporating material from dynamical systems theory and taking a more visual approach. Professor Devaney's coathored *Differential Equations* textbook resulting from that project is now in its fourth edition. He has also produced the Mandelbrot Set Explorer, an online, interactive series of explorations designed to teach students at all levels about the mathematics behind the images known as the Mandelbrot and Julia sets.

In 1994 Professor Devaney received the Award for Distinguished University Teaching from the Northeastern Section of the Mathematical Association of America. In 1995 he was the recipient of the Deborah and Franklin Tepper Haimo Award for Distinguished University Teaching. Professor Devaney received the Boston University Scholar/Teacher of the Year Award in 1996. In 2002 he received a National Science Foundation Director's Award for Distinguished Teaching Scholars as well as the International Conference on Technology in Collegiate Mathematics Award for Excellence and Innovation with the Use of Technology in Collegiate Mathematics. He received Boston University's Metcalf Award for Teaching Excellence in 2003, and in 2004 he was named the Carnegie/CASE Massachusetts Professor of the Year. In 2005 he received the Trevor Evans Award from the Mathematical Association of America for an article entitled "Chaos Rules," published in *Math Horizons*. In 2009 Professor Devaney was inducted into the Massachusetts Mathematics Educators Hall of Fame, and in 2010 he was named the Feld Family Professor of Teaching Excellence at Boston University. ■

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SUPPLEMENTAL MATERIAL

Mastering Differential Equations: The Visual Method

Scope:

The field of differential equations goes back to the time of Newton and Leibniz, who invented calculus because they realized that many of the laws of nature are governed by what we now call differential equations.

A differential equation is an equation involving velocities or rates of change. More precisely, it's an equation for a missing mathematical expression or expressions in terms of the derivatives (i.e., the rates of change) of these expressions. Differential equations arise in all areas of science, engineering, and even the social sciences. The motion of the planets in astronomy, the growth and decline of various populations in biology, the prediction of weather in meteorology, the back-and-forth swings of a pendulum from physics, and the evolution of chemical reactions are all examples of processes governed by differential equations.

In the old days, when we were confronted with a differential equation, the only technique available to us to solve the equation was to find a specific formula for the solution of the equation. Unfortunately, that can rarely be done—most differential equations have no solutions that can be explicitly written down. So in the past, scientists would often come up with a simpler model for the process they were trying to study—a model that they could then solve explicitly. Of course, this model would not be a completely accurate description of the actual physical process, so the solution would only be valid in a limited setting.

Nowadays, with computers (and, more importantly, computer graphics) readily available, everything has changed. While computers still cannot explicitly solve most differential equations, they can often produce excellent approximations to the exact solution. More importantly, computers can display these solutions in a variety of different ways that allow scientists or engineers to get a good handle on what is happening in the corresponding system.

In this course, we take this more modern approach to understanding differential equations. Yes, we grind out the explicit solution when possible, but we also look at these solutions geometrically, plotting all sorts of different graphs that explain the ultimate behavior of the solutions.

In each of the 4 sections of this course, we begin with a simple model and then use that model to introduce various associated topics and applications. No special knowledge of the field from which the model arises is necessary to understand the material. Two broad subthemes appear over and over again in the course: One is the concept of bifurcation—how the solutions of the equations sometimes change dramatically when certain parameters are tweaked just a little bit. The second subtheme is chaos—we will see how extremely unpredictable behavior of solutions occasionally arises in the setting of what would seem to be a relatively simple system of differential equations.

In the first part of the course, we discuss the simplest types of differential equations—first-order differential equations—using several population models from biology as our examples. The simplest model is the unlimited population growth differential equation. Later examples, such as the logistic population growth model (a limited growth model), take into account the possibility of overcrowding and harvesting. Using elementary techniques from calculus, we solve these equations analytically when possible and plot the corresponding qualitative pictures of solutions, such as the slope field, the phase line, and the bifurcation diagram. We also describe a simple algorithm that the computer uses to generate these solutions and show how this technique can sometimes fail (often because of chaos). Finally, we introduce a parameter into our model and see how very interesting bifurcations arise.

In the second part of the course, we turn our attention to second-order differential equations. Our model here is the mass-spring system from physics. We see relatively straightforward behavior of solutions as long as the mass-spring system is not forced, but when we introduce periodic forcing into the picture, the much more complicated (and sometimes disastrous) behaviors known as beating modes and resonance occur.

The next part of the course deals with systems of differential equations. Our model here is the predator-prey model from ecology. At first we concentrate on linear systems.

We see how various ideas from linear algebra allow us to solve and analyze these types of differential equations. Bifurcations recur as we investigate the trace determinant plane. We then move on to a topic of considerable interest nowadays, nonlinear systems. Almost all models that arise in nature are nonlinear, but these are the differential equations that can rarely be solved explicitly. We investigate several different models here, including competitive systems from biology and oscillating chemical reactions. Another model will be the Lorenz system from meteorology; back in the 1960s, this was the first example of a system of differential equations that was shown to exhibit chaotic behavior.

In the final part of the course, we turn to the concept of iterated functions (also called difference equations) to investigate the chaos we observed in the Lorenz system. Our model here is an iterated function for the logistic population model from biology, a very different kind of model than our earlier logistic differential equation. In this case, we see that lots of chaos emerges, even in the simple iterated function, and we begin to understand how we can analyze and comprehend this chaotic behavior.

While calculus is a central notion in differential equations, we will not delve into many of the specialized techniques from calculus that can be used to solve certain differential equations. The only topics from calculus that we presume familiarity with are the derivative, the integral, and the notion of a vector field. Any other relevant concepts from calculus or linear algebra are introduced before being used in the course.

By the end of this course, you will come to see how all the concepts from algebra, trigonometry, and calculus come together to provide a beautiful and comprehensive tool for investigating systems that arise in all areas of science and engineering. You will also see how the field of differential equations is an area of mathematics that, unlike algebra, trigonometry, and calculus, is still developing, with many new and exciting ideas sparking interest across all disciplines.

What Is a Differential Equation?

Lecture 1

An **ordinary differential equation** (ODE) is an equation for a missing function (or functions) in terms of the derivatives of those functions. Recall that the derivative of a function $y(t)$ is denoted either by $y'(\mathrm{t})$ or by dy/dt. Suppose $y(t_0) = y_0$. Then the derivative $y'(t_0)$ gives the slope of the tangent line to the graph of the function $y(t)$ at the point (t_0, y_0).

In the old days, the only tools we had to solve ODEs were analytical methods—a variety of different tricks from calculus that sometimes enabled us to write down an explicit formula for the solution of the differential equation. Unfortunately, most ODEs cannot be solved in this fashion. But times have changed: Now we can use the computer to approximate solutions of ODEs. And we can use computer graphics and other geometric methods to display solutions graphically. So this gives us 2 new ways to solve differential equations, and these are the methods that we will emphasize in this course.

Probably the best-known differential equation (and essentially the first example of a differential equation) is Newton's second law of motion. Drop an object from your rooftop. If y measures the position of the center of mass of the object, then we would like to know its position at time t, that is, $y(t)$. Newton's law tells us that mass times acceleration is equal to the force on the object. So, if m is the mass, then we have $my'' = F(y)$, where F is the force acting on the object when it is in position $y(t)$. So we have a differential equation for $y(t)$.

Another example of a differential equation is the mass-spring system (or harmonic oscillator). Here $y(t)$ is the missing function that measures the position of a mass attached to a spring attached to a ceiling. When we pull the mass down and let it go, $y(t)$ gives us the resulting motion of the mass over time. The function $y(t)$ is determined by the differential equation $y'' + by' + ky = G(t)$.

In this case, we have parameters like b and k as well as a forcing term $G(t)$ that depends only on t. We will see very different behaviors for the mass-spring system depending on these parameters and the forcing term. Sometimes a small change in one of these parameters creates a major change in the behavior of solutions. Such phenomena are called **bifurcations**, one of the subthemes we will encounter often in this course.

Most of our course will be spent considering systems of differential equations. Systems of ODEs involve more than one missing function in the equation. We will consider numerous such examples, but the most famous is undoubtedly the Lorenz system from meteorology, below.

$$\begin{aligned}
x' &= -10x + 10y \\
y' &= -xz + Rx - y \\
z' &= xy - \tfrac{8}{3}z
\end{aligned}$$

The Lorenz system of equations was the first example of a system of ODEs that exhibits chaotic behavior, another subtheme that we will encounter throughout the course.

Let's look at our first differential equation: the unlimited population growth model from biology, which is perhaps the simplest nontrivial differential equation. Suppose we have a species living in isolation (with no predators, no overcrowding, and no emigration) and we want to predict its population as a function of time. Call the population $y(t)$.

Our assumption is that the rate of growth of the population is directly proportional to the current population. This translates to the ODE $y' = ky$. Here k is a constant (a parameter) that depends on which species we are considering. This is an example of a first-order ODE, since the equation depends on at most the first derivative of $y(t)$. Usually we wish to find the solution of an **initial value problem**, that is, a specific solution of the ODE that satisfies $y(0) = y_0$ where y_0 is the given initial population.

For simplicity, suppose $k = 1$, so our differential equation is $y' = y$. One solution is the exponential function $y(t) = e^t$, since the derivative of the

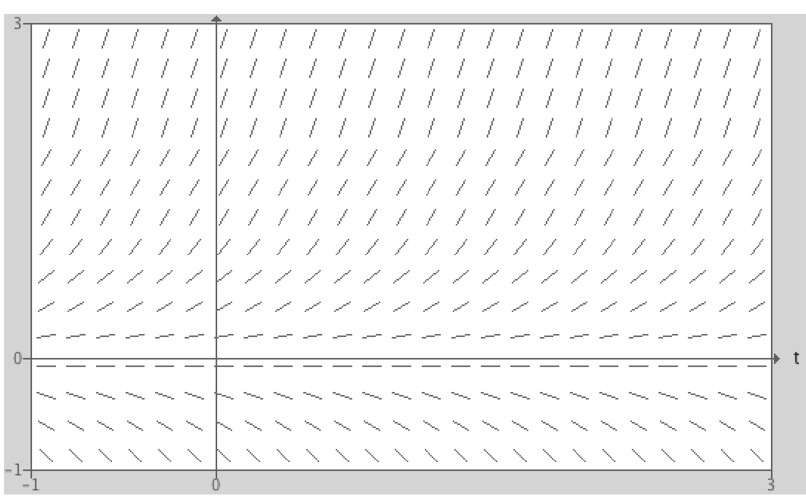
function e^t is e^t (so that y' does indeed equal y). Another solution is $y(t) = Ce^t$, where C is some constant. Note that when $C = 0$, the solution is the constant function $y(t)$ identically equal to zero. This is an **equilibrium solution**, one of the most important types of solutions of differential equations. The **general solution** of the equation is $y(t) = Ce^t$, since we can solve any initial value problem using this formula. That is, given $y(0) = y_0$, the solution satisfying this initial condition is given by setting $C = y_0$.

More generally, we can consider the equation $y' = ky$, where k is some constant, say $k = 2$. As before, other solutions are of the form Ce^{2t}, where C is an arbitrary constant. Again we see that the solution with $C = 0$ is an equilibrium solution. We can solve any initial value problem by choosing C as our initial value $y(0)$, so Ce^{2t} is the general solution of this differential equation.

Here are 2 of the simplest methods for visualizing solutions of differential equations.

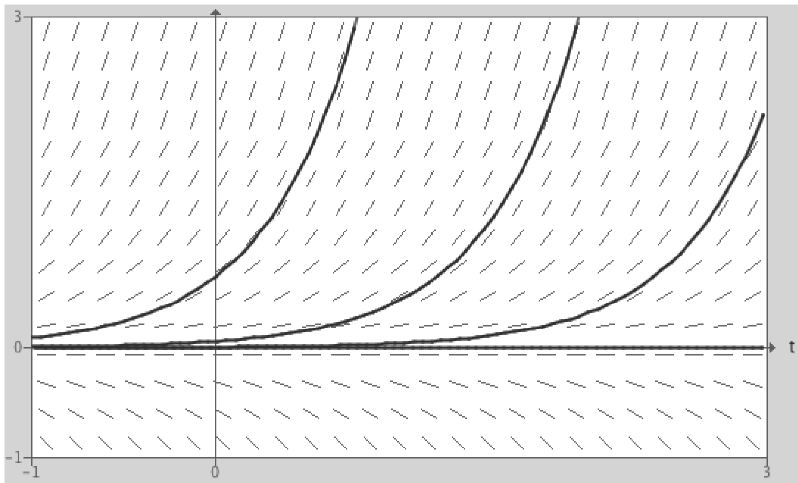
1. The right-hand side of the differential equation tells us the slope of the solution of the ODE at any time t and population y. So in the t-y plane, we draw a tiny straight line with slope equal to the value on the right-hand side. A collection of such slopes is the slope field below.

Figure 1.1



2. Then a solution must be everywhere tangent to the slope field, so we can sketch in the graphs of our solutions.

Figure 1.2



Note the constant solution, $y(t) = 0$. This is our equilibrium solution.

Important Terms

bifurcation: A major change in the behavior of a solution of a differential equation caused by a small change in the equation itself or in the parameters that control the equation. Just tweaking the system a little bit causes a major change in what occurs.

equilibrium solution: A constant solution of a differential equation.

general solution: A collection of solutions of a differential equation from which one can then generate a solution to any given initial condition.

initial value problem: A differential equation with a collection of special values for the missing function such as its initial position or initial velocity.

ordinary differential equation (ODE): A differential equation that depends on the derivatives of the missing functions. If the equation depends on the partial derivatives of the missing functions, then that is a partial differential equation (PDE).

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap 1.1.

Guckenheimer and Holmes, *Nonlinear Oscillations*.

Hirsch, Smale, and Devaney, *Differential Equations*, chap 1.1.

Roberts, *Ordinary Differential Equations*, chap 1.1–1.3.

Strogatz, *Nonlinear Dynamics and Chaos*, chap 1.1.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, HPG Solver.

Problems

1. Let's review some ideas from calculus that we used in this lecture.

a. Compute the derivative of $y(t) = t^3 + e^t$.

b. Compute the second derivative of $y(t) = t^3 + e^t$.

c. Find a function $f(t)$ whose derivative is $t^3 + e^t$.

d. Sketch the graph of the exponential function $y(t) = e^t$.

e. Find the solution of the equation $e^{(2t)} = 1$.

2. Sketch the slope field and solution graphs for the differential equation $y' = 1$.

3. What are some solutions of the differential equation in problem 2?

4. Repeat problems 2 and 3 for $y' = t$.

5. In the unlimited population growth model $y' = ky$, what happens to solutions if k is negative?

6. Find the general solution of the differential equation $y' = t$.

7. Find all of the equilibrium solutions of the differential equation $y' = y^2 - 1$. Also plot the slope field. What do you think will happen to solutions that are not equilibria?

8. What would be the general solution of the simple differential equation $y' = 0$? What is the behavior of all of these solutions?

9. Consider the differential equation given by Newton's second law, $my'' = g$, where we assume that m and g are constants. Can you find some solutions of this equation?

Exploration

Think about the fact that back in the 1600s, Isaac Newton was able to come up with not only the second law of motion in physics but also the basics of calculus and differential equations (plus everything else he discovered in the sciences). He was quite an intellectual! There is plenty to read about his life and discoveries on the web. For an introduction, go to http://www.newton.ac.uk/newtlife.html.

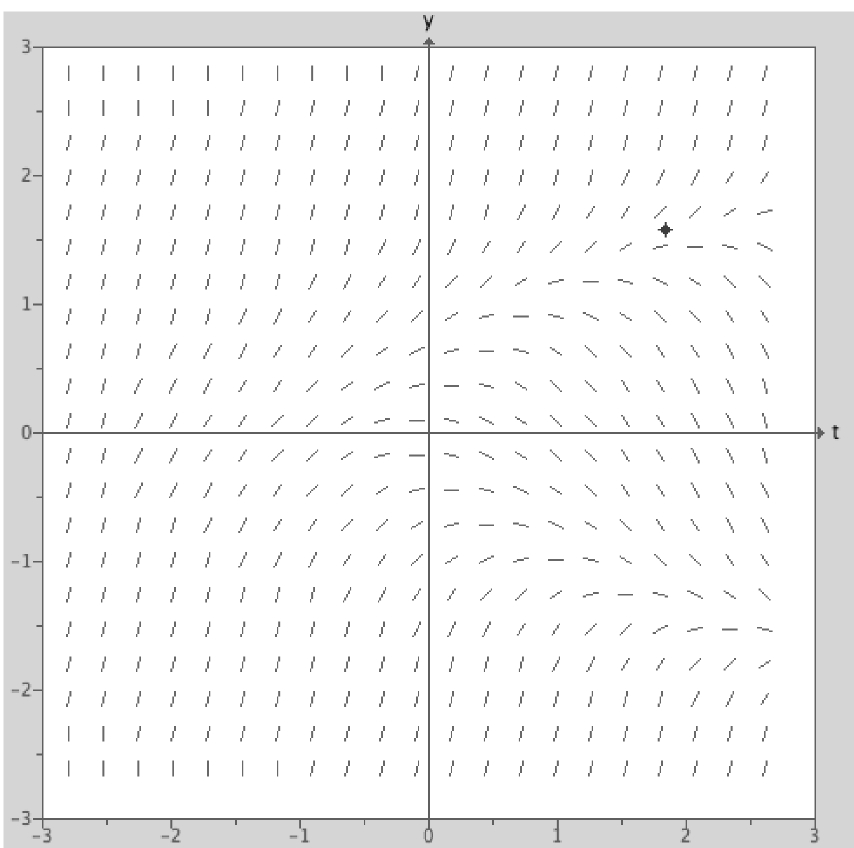
A Limited-Growth Population Model

Lecture 2

We begin this lecture by investigating a more complicated (but more realistic) population model, the limited-growth population model (also known as the logistic population growth model). The corresponding slope field gives us an idea of how solutions will behave, at least from a qualitative point of view.

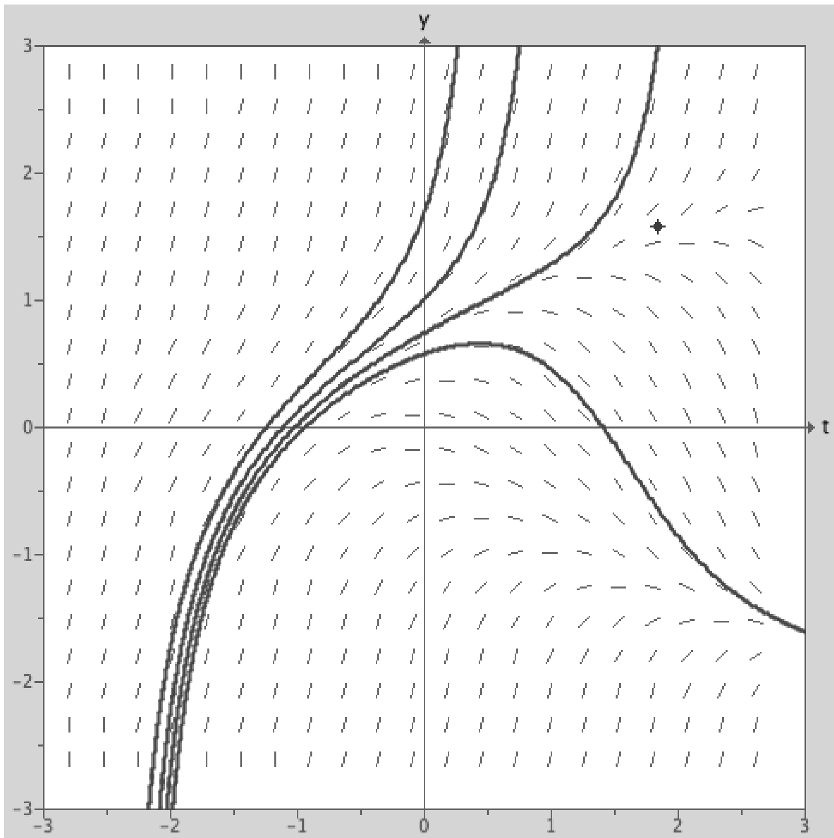
This does not always happen, however. Look at the slope field for the differential equation $y' = y^2 - t$, and put a little target at (2, 2). Can you see where a solution in the lower left quadrant should begin so that it hits the given target?

Figure 2.1



Well, maybe not. Nearby solutions tend to veer off from one another as y and t increase, as can be seen from the slope field, so the slope field does not tell us everything in this case.

Figure 2.2



For the limited population growth model, we will assume that overcrowding may occur, which will hinder population growth or lead to population decline (if the population is too large). We make 2 assumptions about our population, $y(t)$:

- If $y(t)$ is small, the rate of population growth is proportional to the current population (as in our unlimited growth model).

- There is a **carrying capacity** N (an "ideal" population size) such that

  $y(t) > N$ means that $y(t)$ decreases ($y'(t) < 0$), and

  $y(t) < N$ means that $y(t)$ increases ($y'(t) > 0$).

The first assumption tells us that $y'(t) = ky$ if $y(t)$ is small. So we let our differential equation assume the form

$$y' = ky(?),$$

where the expression (?)

1. is approximately equal to 1 if y is small;
2. is negative if $y > N$;
3. is positive if $y > N$; and
4. is 0 if $y = N$.

Then one possibility (the simplest) for the term (?) is $(1 - y/N)$.

This yields the limited population growth ODE

$$\frac{dy}{dt} = ky(1 - y/N).$$

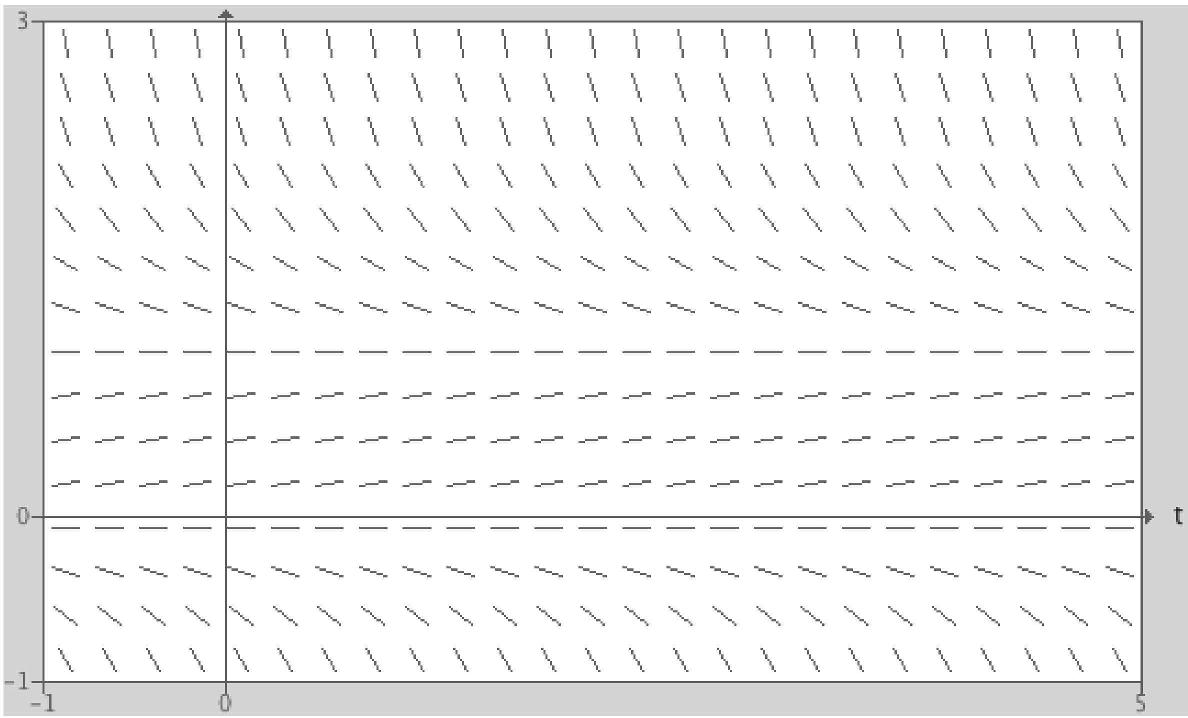
For simplicity, let's consider the case where $k = N = 1$:

$$\frac{dy}{dt} = y(1 - y).$$

This doesn't mean our carrying capacity population is just one individual (that would not be a very interesting environment). Rather, think of $y(t)$ as measuring the percentage of the carrying capacity. We will solve this differential equation analytically in a later lecture, but for now we will use qualitative methods to view the solutions.

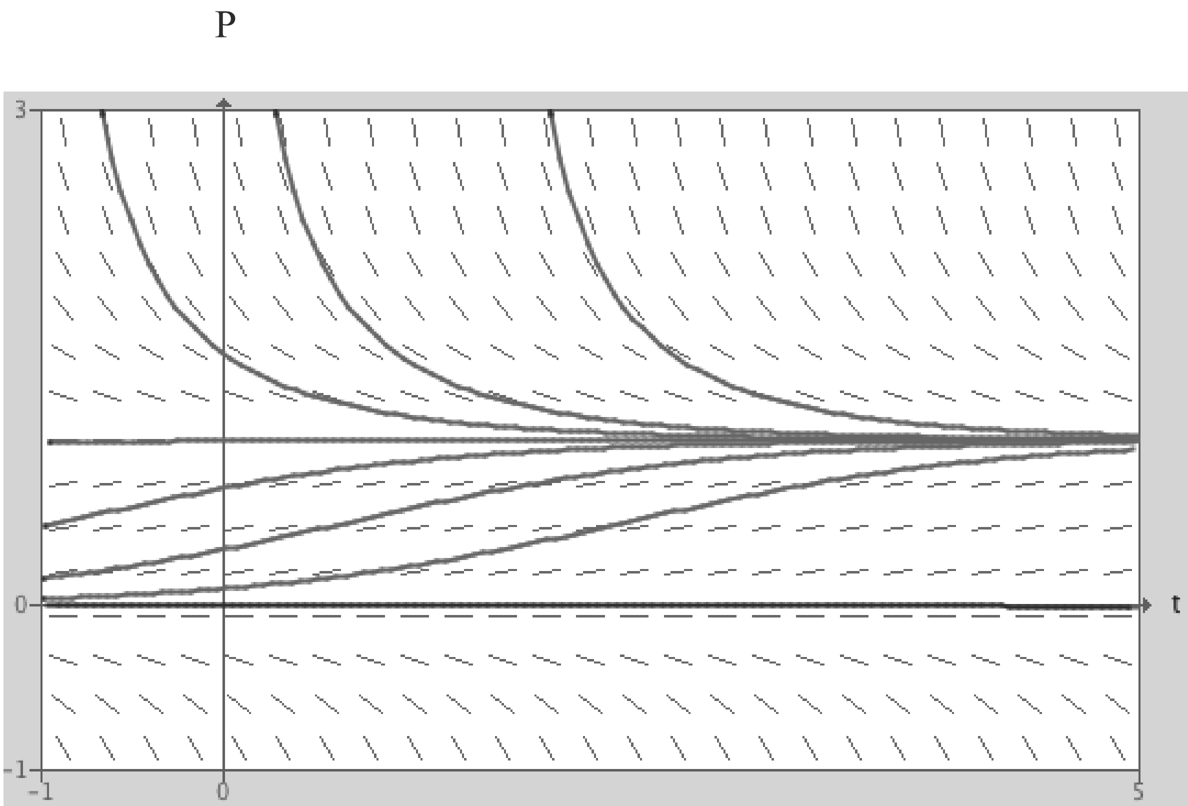
Here is the slope field for the limited population growth model equation. Note the horizontal slopes when $y = 0$ or $y = 1$; these are our equilibrium points.

Figure 2.3



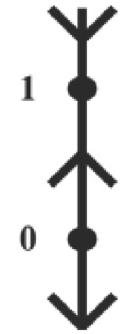
Here are some solution graphs. Note that they do exactly as we expected; they all tend to the equilibrium $y = 1$, the carrying capacity (assuming, of course, that the population is nonzero to start).

Figure 2.4



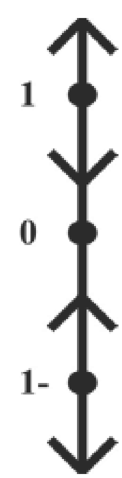
With an eye toward what comes later—when we deal with differential equations whose solutions live in higher dimensional spaces—we will plot the **phase line** for this ODE. This is a picture of the motion of a particle along a straight line where the position of the particle at time t is the value of $y(t)$.

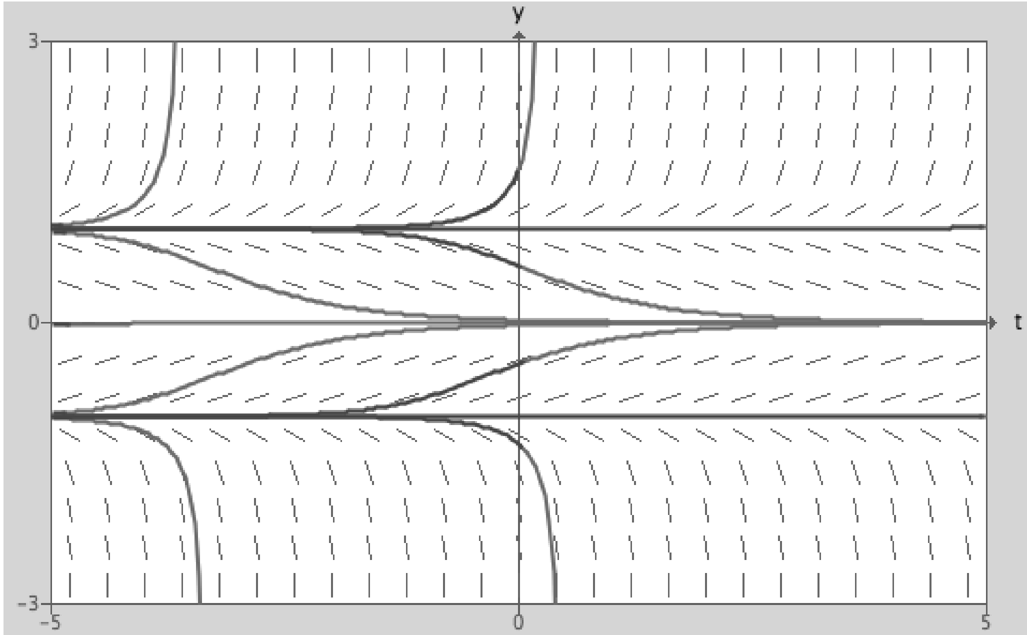
Figure 2.5



This course deals primarily with autonomous differential equations. These are ODEs of the form $y' = F(y)$ (i.e., the right-hand side does not depend on t). We simply find all the equilibrium points by solving $F(y) = 0$. Between 2 such equilibria, the slopes are either always positive or always negative (assuming $F(y)$ is a continuous function), so solutions either always increase or always decrease between the equilibrium solutions. For example, for the ODE $y' = y^3 - y$, we have 3 equilibria at $y = 0$, 1, and -1. Above $y = 1$, we have $y' > 0$. Between $y = 0$ and y = 1, we have $y' < 0$. Between $y = -1$ and $y = 0$, we have again $y' > 0$. And below $y = -1$, we have $y' < 0$. So we know what the solutions will do, at least qualitatively. The phase line and some representative solution graphs are shown below.

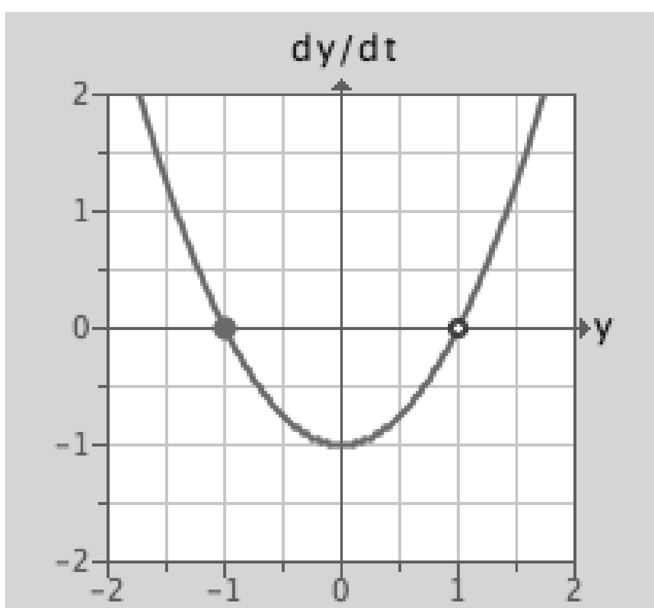
Figure 2.6





We can use the graph of the right-hand side of the differential equation to read this behavior off. Consider $y' = y^2 - 1$. Here is the graph of $dy/dt = F(y)$.

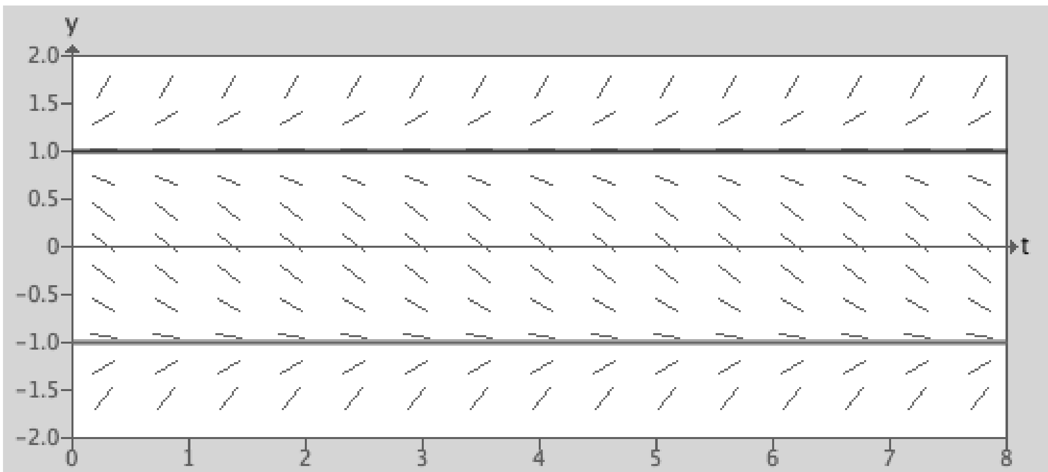
Figure 2.7



A glance at this graph tells us that if $y > 1$ or $y < -1$, then $y' = y^2 - 1 > 0$, so solutions must increase in this region. If $-1 < y < 1$, then we have $y' < 0$, so solutions decrease. And if $y = 1$ or $y = -1$, we have equilibrium points.

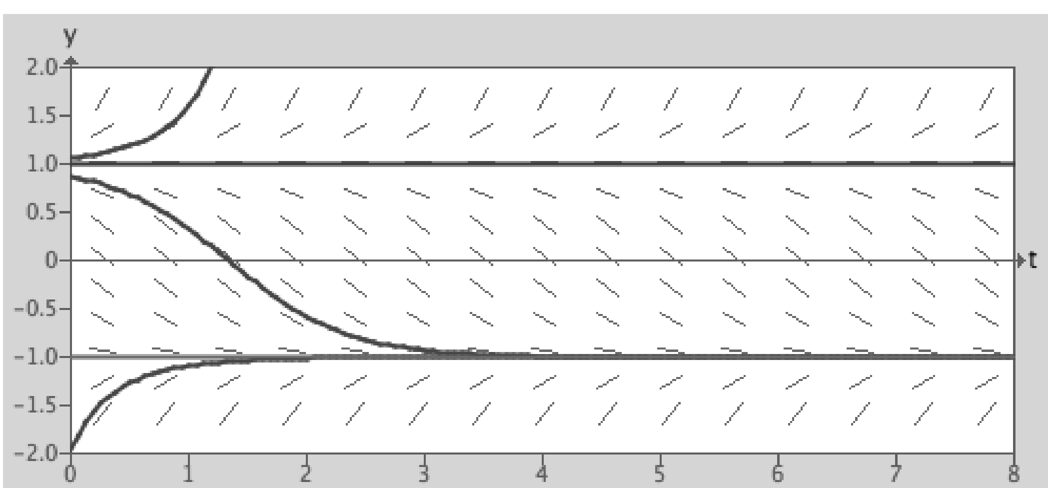
With this information, we know that the slope field looks as follows.

Figure 2.8



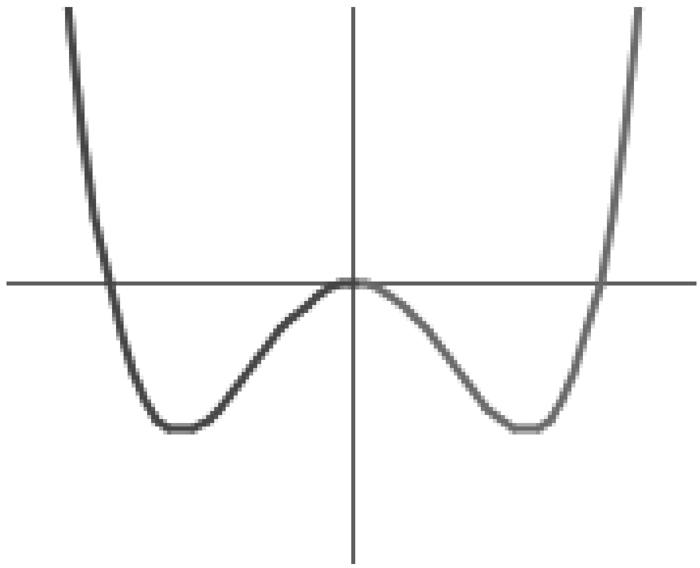
And our solutions behave as follows.

Figure 2.9



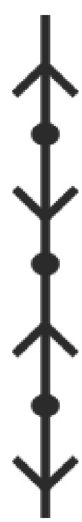
As another example, for the ODE $y' = y^4 - y^2 = y^2(y^2 - 1)$, we see that there are 3 equilibrium points: at 0, 1, and −1. The graph of the expression $F(y) = y^2(y^2 - 1)$ looks as follows.

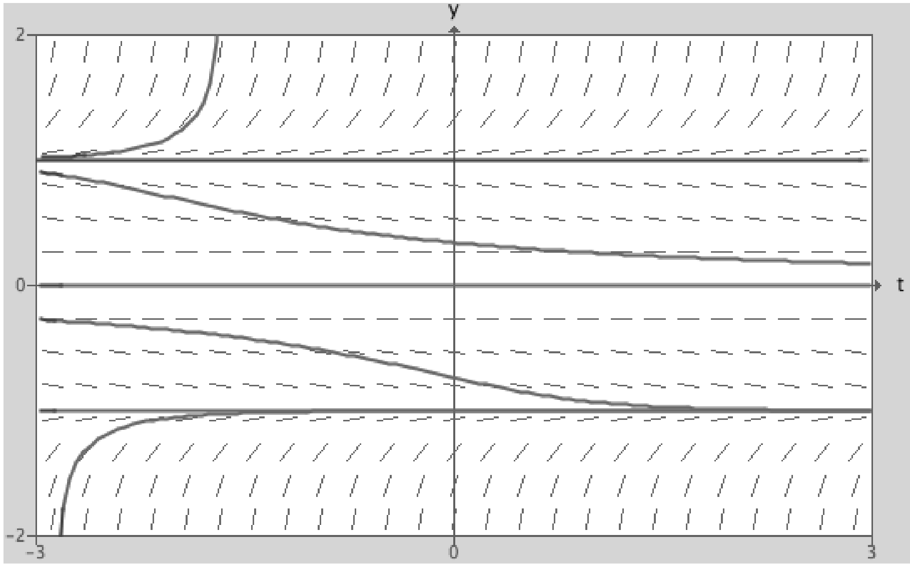
Figure 2.10



This graph tells us what happens between the equilibria: We have $y' > 0$ if $|y| > 1$, while $y' < 0$ if $0 < |y| < 1$. So the phase line and the solution graphs are as follows.

Figure 2.11





Note that there are 3 different types of equilibrium points for this ODE. The equilibrium point at $y = 1$ is called a **source** since all nearby solutions move away from this equilibrium. The equilibrium point at $y = -1$ is called a **sink** since all nearby solutions move closer to this point. And the equilibrium point at $y = 0$ is called a **node** since it is neither a sink nor a source.

Important Terms

carrying capacity: In the limited population growth population model, this is the population for which any larger population will necessarily decrease, while any smaller population will necessarily increase. It is the ideal population.

node: An equilibrium solution of a first-order differential equation that has the property that it is neither a sink nor a source.

phase line: A pictorial representation of a particle moving along a line that represents the motion of a solution of an autonomous first-order differential equation as it varies in time. Like the slope field, the phase line shows whether equilibrium solutions are sinks or sources, but it does so in a simpler way that lacks information about how quickly solutions are

increasing or decreasing. The phase line is primarily a teaching tool to prepare students to make use of phase planes and higher-dimensional phase spaces.

sink: An equilibrium solution of a differential equation that has the property that all nearby solutions tend toward this solution.

source: An equilibrium solution of a differential equation that has the property that all nearby solutions tend away from this solution.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap 1.3.

Guckenheimer and Holmes, *Nonlinear Oscillations*.

Hirsch, Smale, and Devaney, *Differential Equations*, chap 1.2.

Roberts, *Ordinary Differential Equations*, chaps. 2.1 and 2.3.

Strogatz, *Nonlinear Dynamics and Chaos*, chap 2.1.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, First Order Solutions, HPG Solver, Phase Lines, Target Practice.

Problems

1. Let's begin with some ideas from calculus.

a. What is the integral (antiderivative) of the function $y(t) = t^2 + t$?

b. Solve the equation $y^4 - 4y^2 = 0$.

c. For which values of t is the function $y(t) = t^3 - t$ positive?

d. Sketch the graph of $y(t) = t^2 - 1$.

e. Sketch the graph of $y(t) = t(2 - t)$.

2. Sketch the phase line for $y' = 1$.

3. Sketch the phase line for $y' = -y$.

4. What are the equilibria for the previous differential equation?

5. What are the equilibrium points for $y' = -1$?

6. What is the phase line for the differential equation $y' = y^n$, where n is some positive integer? Does the answer depend on n?

7. What is the phase line for the differential equation $y' = y^2(1 + y)$?

8. Find all equilibrium points for the differential equation $y' = \sin(y)$ and then sketch the phase line.

9. Find an example of a differential equation that has equilibria at each integer value and each of these equilibria is a node.

10. Find an example of a differential equation that has exactly 2 equilibrium points, each of which is a node.

Exploration

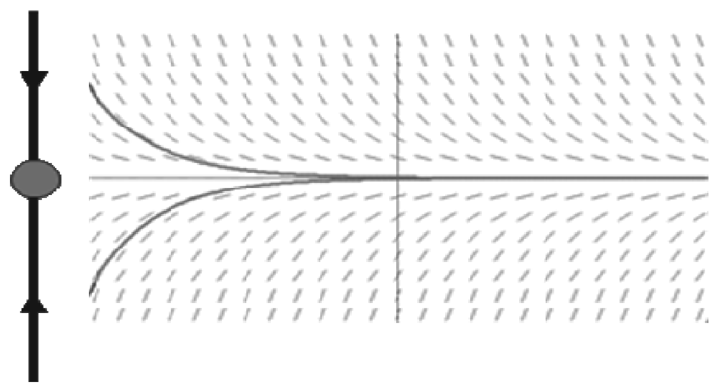
Consider the differential equation $y' = y^2$. We know what the slope field and phase line look like, but can you find actual solutions? How about for $y' = |y|$? Notice that the phase lines are the same for these differential equations, but the solutions are very different. Can you find other differential equations that have the same phase lines but different solutions?

Classification of Equilibrium Points

Lecture 3

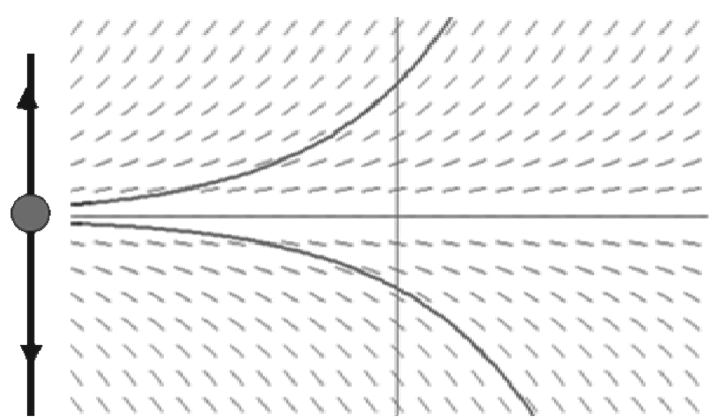
In this lecture, we see that slope field allows us to completely understand the behavior of solutions of first-order, autonomous differential equations. Recall that there are 3 different types of equilibria: sinks, sources, and nodes.

Figure 3.1



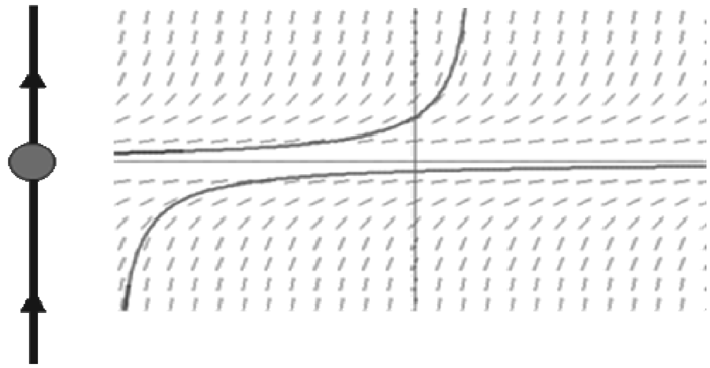
sink

Figure 3.2



source

Figure 3.3



node

There are actually lots of different things that can happen when an equilibrium point is a node. For example, the differential equation $y' = y^2$ has a node at $y = 0$. But for any nonzero value of y, solutions tend away from 0 if the initial value $y_0 > 0$, whereas solutions tend to 0 if $y_0 < 0$. The differential equation $y' = 0$ also has an equilibrium point at 0, and this equilibrium point is also a node, since all solutions are equilibria. And there are examples where an infinite collection of equilibrium points can accumulate on a given equilibrium point, again giving us a node.

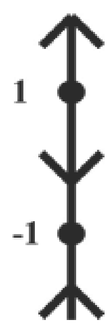
We can use calculus to determine whether a given equilibrium point y_0 for the ODE $y' = F(y)$ is a sink, a source, or a node. Since y_0 is an equilibrium point, we know that the graph of F crosses the y-axis at y_0. If the graph of F is increasing as it passes through y_0, then y' is positive when $y > y_0$ and y' is negative when $y < y_0$. This says that solutions move away from y_0 whenever y is close to y_0, so y_0 is a source. On the other hand, if the graph of F is decreasing through y_0, then similar arguments show that y_0 is a sink.

But we know from calculus that if $F'(y_0) > 0$, then $F(y)$ is increasing, and if $F'(y_0) < 0$, then $F(y)$ is decreasing. So this gives us the **first derivative test for equilibrium points**: Suppose we have the differential equation $y' = F(y)$, and y_0 is an equilibrium point.

- If $F'(y_0) > 0$, then y_0 is a source.
- If $F'(y_0) < 0$, then y_0 is a sink.
- If $F'(y_0) = 0$, then we get no information.

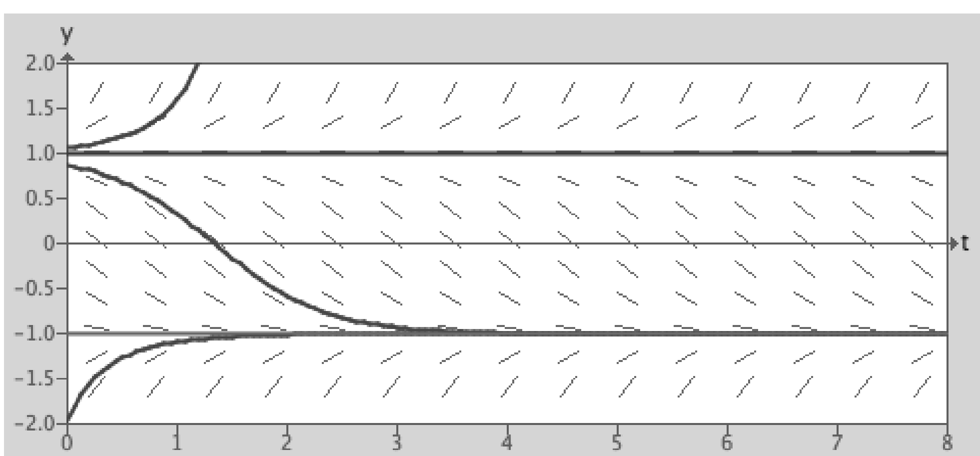
Consider $y' = F(y) = y - y^2$. If $y = 1$ or $y = 0$, we have equilibrium points. Moreover, $F'(y) = 1 - 2y$. This means that $F'(1) = -2 < 0$, so 1 is a source. Similarly, $F'(0) = 1$, so 0 is a sink. Therefore we know that the phase line is as follows.

Figure 3.4



And our solutions behave as follows.

Figure 3.5



As another example, consider $y' = y^3 - y$. We can factor the right-hand side into $y' = y(y + 1)(y - 1)$, so we have equilibrium points at $y = 0$, 1, and -1. Because $F'(y) = 3y^2 - 1$, we know that

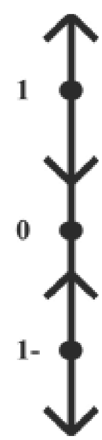
$F'(0) = -1$, so 0 is a sink;

$F'(1) = 2$, so 1 is a source; and

$F'(-1) = 2$, so -1 is also a source.

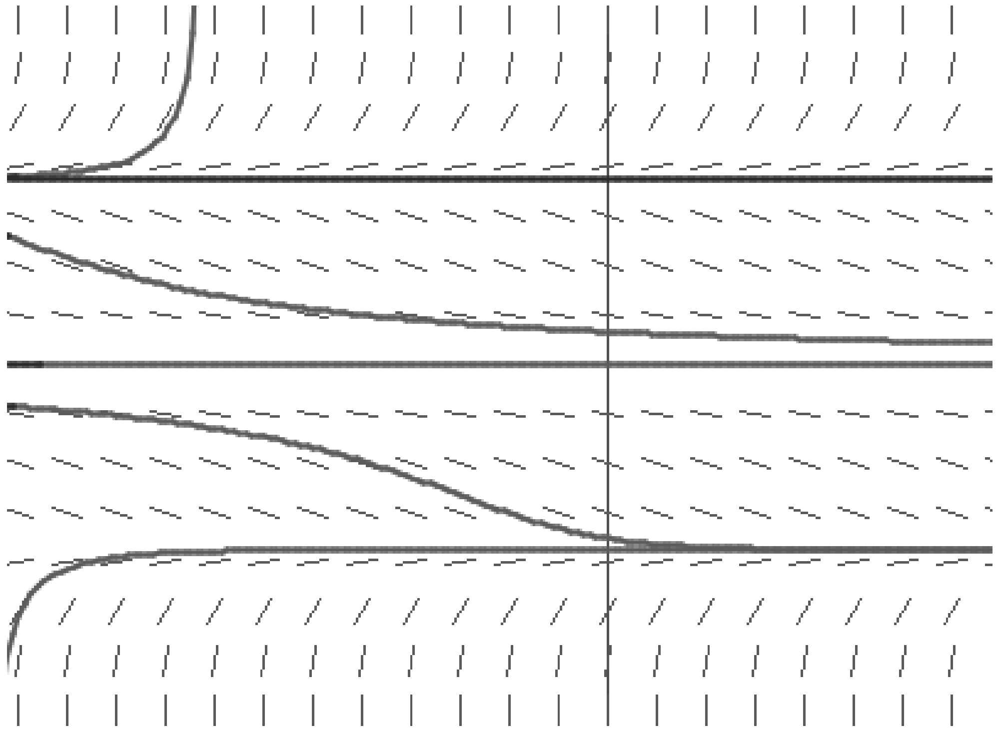
The phase line is therefore as follows, and we again know the behavior of all solutions.

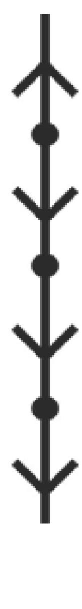
Figure 3.6



Let's take another example: $y' = y^4 - y^2$. We have equilibrium points at 0 and at ± 2. Since $F'(y) = 4y^3 - 2y$, we have $F'(1) = 2 > 0$; so 1 is a source. Because $F'(-1) = -2$, -1 is a sink. The derivative at the 0 vanishes, so we get no information. But what we have about the 2 equilibria at ± 1 is enough to tell us that 0 is a node.

Figure 3.7





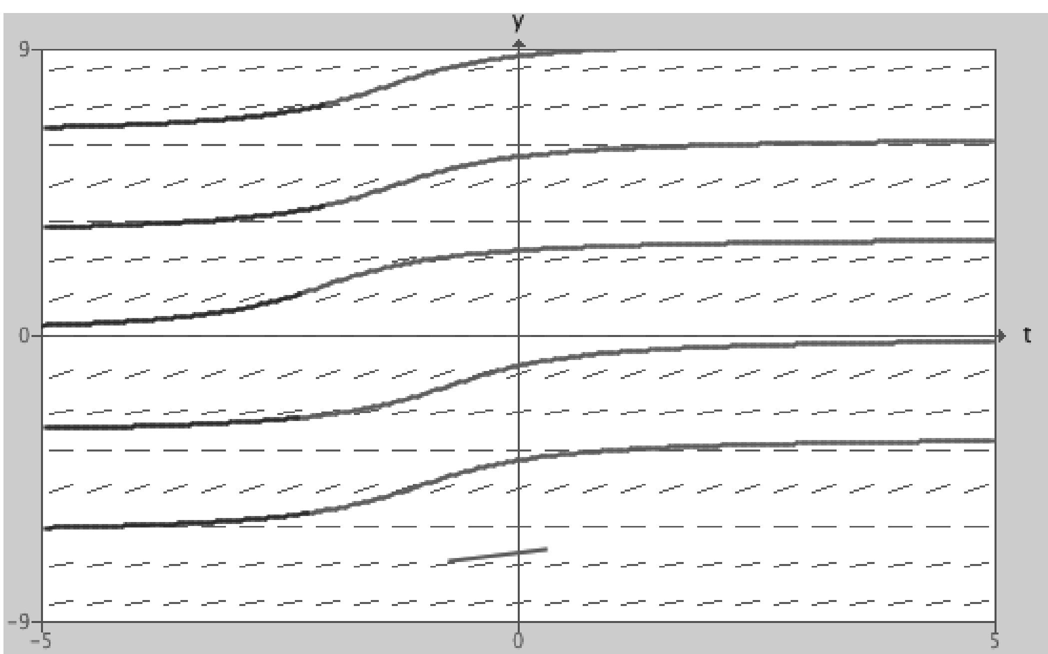
We now turn to perhaps the most important theorem regarding first-order differential equations, the existence and uniqueness theorem. Suppose we have a differential equation $y' = F(y)$, and suppose the function $F(y)$ is nice at the point y_o. (For our purposes, "nice" means that the expression is

continuously differentiable in y at the given point y_0.) Then the theorem states that

- there is a solution to the ODE that satisfies y(t0) = y0, and
- that this is the only solution that satisfies y(t0) = y0.

Let's work a couple of examples. First, $y' = \sin^2(y)$. We have equilibrium points at $y = 0, \pm\pi, \pm 2\pi, \ldots$. At all other points, the slope field is positive. So all other solutions are always increasing. However, any solution that starts between $y = 0$ and $y = \pi$ can never cross these 2 lines. The solution is trapped in this region, and since it is always increasing, the solution must tend to $y = 0$ in backward time and to $y = \pi$ in forward time. We have similar behavior between any 2 adjacent equilibrium points.

Figure 3.8

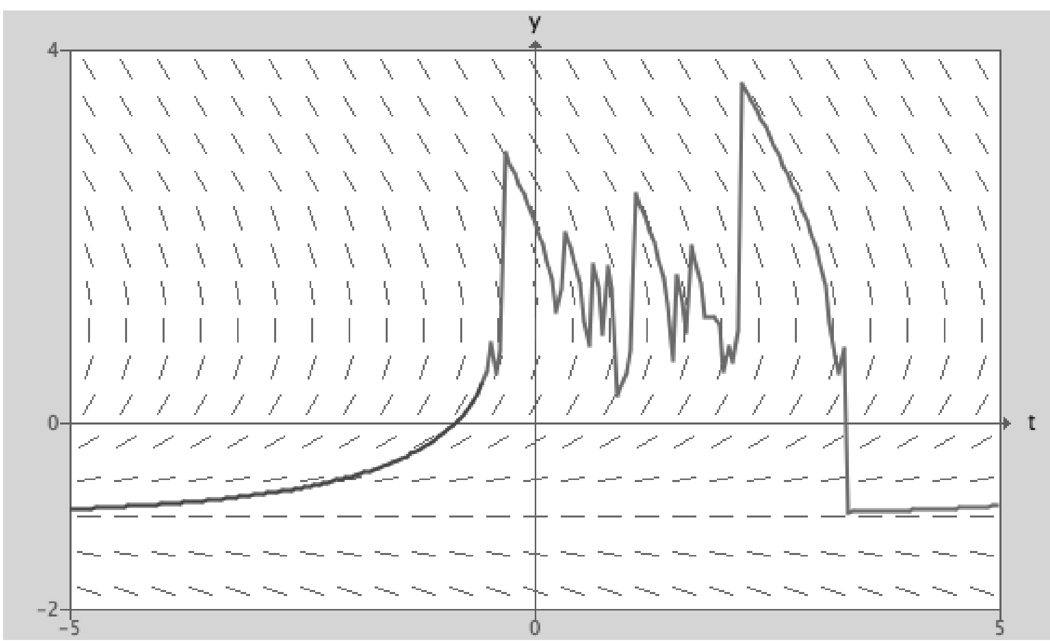


Now let's consider $y' = y/t$. Here we have problems when $t = 0$. The right-hand side of the ODE is not even defined at $t = 0$, never mind continuously differentiable there. It is easy to check that $y(t) = kt$ is a solution of this ODE for any value of k. For each such k, we have $y(0) = 0$, so we now have infinitely many solutions that satisfy $y(0) = 0$. But there are no solutions that satisfy $y(0) = y_0$ for any nonzero value of y_0. Even worse, when the right-hand side of the ODE is not nice, the numerical methods we are using to plot solutions may fail. For example, the differential equation

$$y' = \frac{1+y}{1-y}$$

is not defined at $y = 1$. Look at what happens when we try to use the computer to plot the corresponding solution.

Figure 3.9



Something is clearly wrong here; the numerical method the computer is using has failed. We will describe why this happens in Lecture 6.

Important Term

first derivative test for equilibrium points: This test uses calculus to determine whether a given equilibrium point is a sink, source, or node. Basically, the sign of the derivative of the right-hand side of the differential equation makes this specification; if it is positive, we have a source; negative, a sink; and zero, we get no information.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 1.5 and 1.6.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 7.2.

Roberts, *Ordinary Differential Equations*, chap. 2.2.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 2.5.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, HPG Solver, Phase Lines.

Problems

1. First let's do some calculus review problems.

a. For which values of t is the function $y(t) = 1 + t^2$ increasing?

b. For which values of t is the function $y(t) = |t|$ decreasing?

c. What are the roots of the function $y(t) = t^2 + 2t + 1$?

d. Where is the previous function increasing?

e. Sketch the graph of $y(t) = t^2 + 2t + 1$.

f. Sketch the graph of $y(t) = \sin(t)$.

g. Compute the derivative of $\cos(3t + 4)$.

2. Find the equilibrium points for $y' = y + 1$, and determine their type.

3. Find the equilibrium points for $y' = -y^2$, and determine their type.

4. Sketch the phase line for $y' = -y^2$.

5. Find the equilibrium points for $y' = y^2 - 1$ and determine their type.

6. Classify the types of equilibrium points for $y' = y^3 - 1$.

7. Does the existence and uniqueness theorem apply to the differential equation $y' = |y|$?

8. Consider $y' = y^2 - A$, where A is a parameter. Find all equilibrium points and determine their type (your answer will, of course, depend on A).

9. Consider $y' = Ay(1 - y)$, where A is a parameter. Find all equilibrium points and determine their type.

10. What is the actual behavior of solutions of

$$y' = \frac{1+y}{1-y}\,?$$

Exploration

We saw that the differential equation in problem 10 causes problems when we try to solve it numerically. Can you come up with other examples of first-order ODEs that break the numerical algorithm your computer uses to solve them?

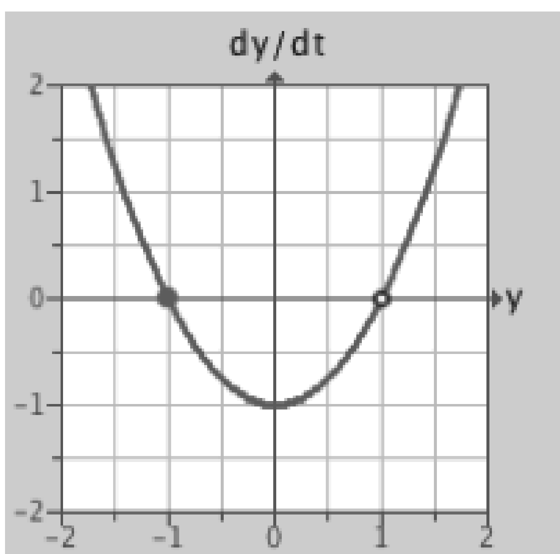
Bifurcations—Drastic Changes in Solutions

Lecture 4

We now introduce the concept of bifurcation, one of the subthemes in this course that will arise over and over. To bifurcate basically means to split apart or to go in different directions. In the theory of differential equations, to bifurcate means to have solutions suddenly veer off in different directions. In order to have such changes, our differential equation must depend on some sort of parameter. Sometimes, when we change this parameter just a little, we find a major change in the behavior of solutions; this is a bifurcation.

Let's start with a simple example: $y' = y^2 + A$. We know that for $A > 0$ there are no equilibrium points; for $A = 0$ there is 1 equilibrium point, a node; and for $A < 0$ there are 2 equilibrium points, at $\pm(-A)^{1/2}$. Because $F'(y) = 2y$, by first derivative test, the positive equilibrium point is a source and the negative is a sink. Look at the function graphs.

Figure 4.1



We see a bifurcation at $A = 0$.

In our limited population growth model

$$y' = ky(1 - y/N)$$

we have 2 parameters: the growth constant k and the carrying capacity N. If we vary these parameters, not much happens to our solutions (as long and both k and N remain positive). Now let's introduce a new parameter into this model to account for harvesting. Suppose our population is a fish population that is subject to a certain amount of fishing governed by the amount of fishing licenses that have been issued. We assume that the rate of growth of this fish population goes down depending on the amount of licenses issued, which is essentially the harvesting rate. Let h be this harvesting rate, so $h \geq 0$. Then one differential equation that illustrates this is the **limited population growth model with harvesting**:

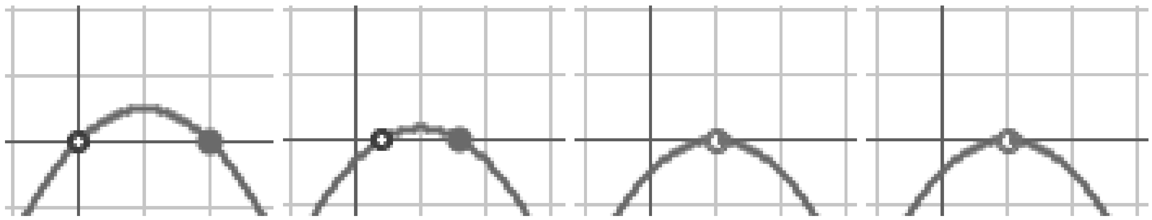
$$y' = ky(1 - y/N) - h.$$

For simplicity, let's revert to our easier model:

$$y' = y(1 - y) - h.$$

First let's find the equilibrium solutions of this equation. Look at the graph of the right-hand side of the ODE (i.e., the graph of $y(1 - y) - h$) to see where this graph crosses the y-axis. When $h = 0$, there are only 2 places where $y' = 0$, namely $y = 0$ and $y = 1$. But as h increases, the graphs move downward and the 2 roots move closer together. At some h-value, they merge into a single root. Then, for slightly higher values of h, there are no roots at all. So our equilibrium points have disappeared. This is the drastic change in the behavior of solutions—this is our bifurcation point.

Figure 4.2



Two questions arise: When does this bifurcation occur? And what are the ramifications in terms of the solutions of our equation (the fate of the fish population)? To answer the first question, we must find the h-value for which our equation

$$y(1-y)-h=0$$

has a single root. Rewriting this equation as

$$-y^2+y-h=0$$

allows us to solve it using the quadratic formula. We find that the roots are given by

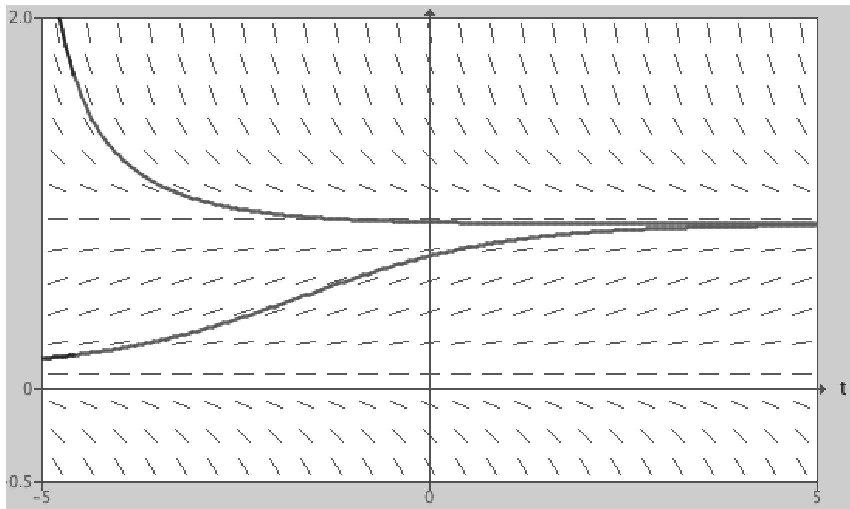
$$q_{\pm}=\frac{1\pm\sqrt{1-4h}}{2}.$$

So there is a unique root when h = 1/4. In this case, we get a single equilibrium point, namely y = 1/2. That is, at h = 1/4, our population levels off at exactly half the carrying capacity. When h < 1/4, we have a pair of equilibria at the 2 points q_+ and q_-. But when h > 1/4, these equilibria have disappeared.

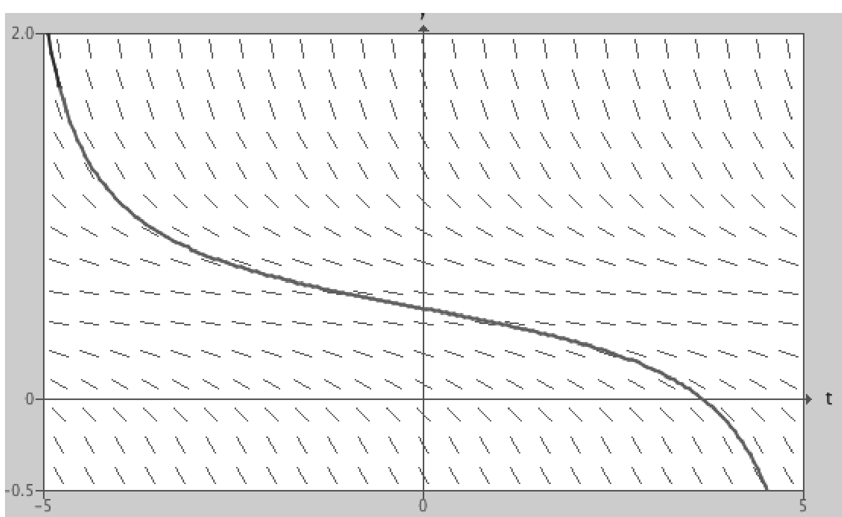
By looking at the graphs above, we see that the equilibrium point given by q_+ is a sink when h < 1/4, while q_- is a source. Looking at the solution graphs, we see that all solutions that start out above the value q_- tend to the equilibrium solution at q_+. So as long as the population is high enough to begin with, harvesting at this rate is fine—the population survives. Even

when $h = 1/4$, all is fine as long as our initial population starts out above the equilibrium point. But as soon as h goes below 1/4, disaster strikes: The population goes extinct.

Figure 4.3



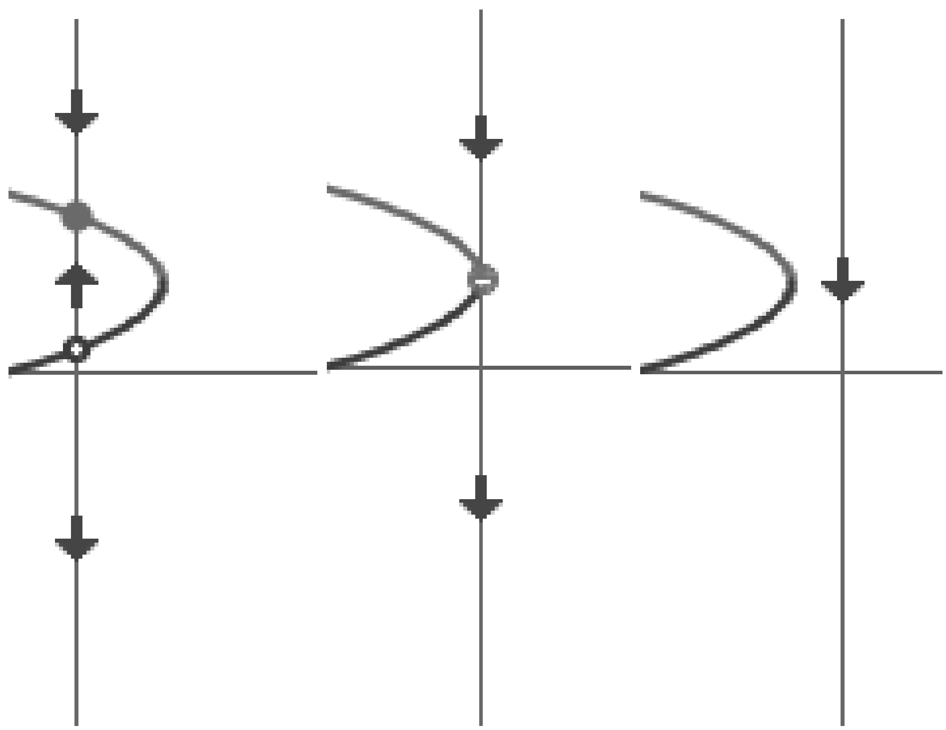
$h < 1/4$; population survives



$h > 1/4$; population dies out

To summarize all of this, we can record all of our phase lines in a **bifurcation diagram**. In this plot, we let the harvesting parameter h lie along the horizontal axis. Over each h-value, we superimpose the corresponding phase line as well as all possible equilibrium points. This enables us to see the dramatic bifurcation that occurs when h passes through 1/4.

Figure 4.4

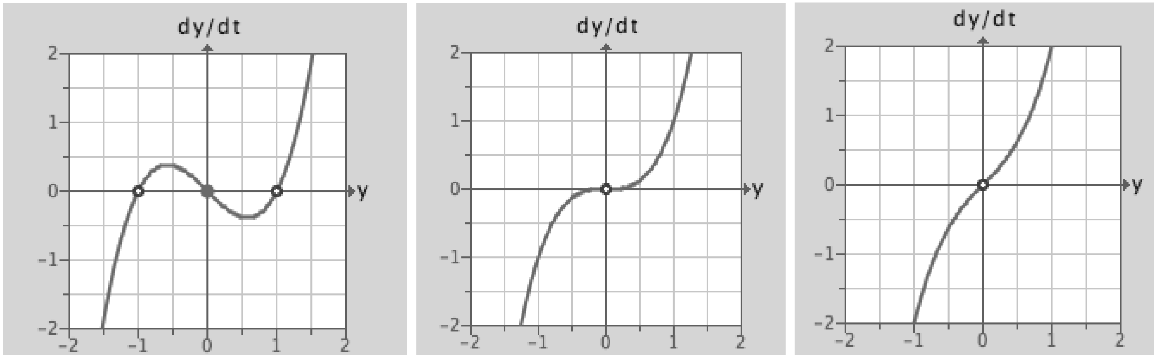


$h < 1/4$ $\quad$ $h = 1/4$ $\quad$ $h > 1/4$

This type of bifurcation is called a **saddle-node bifurcation**.

As another example of a bifurcation, consider the differential equation $y' = ay + y^3$. This equation has equilibrium points at $y = 0$ and $y = \pm\sqrt{-a}$. So there is only one equilibrium point when $a \geq 0$, but there are 3 equilibria when $a < 0$. We see this easily from the graphs of $ay + y^3$ below.

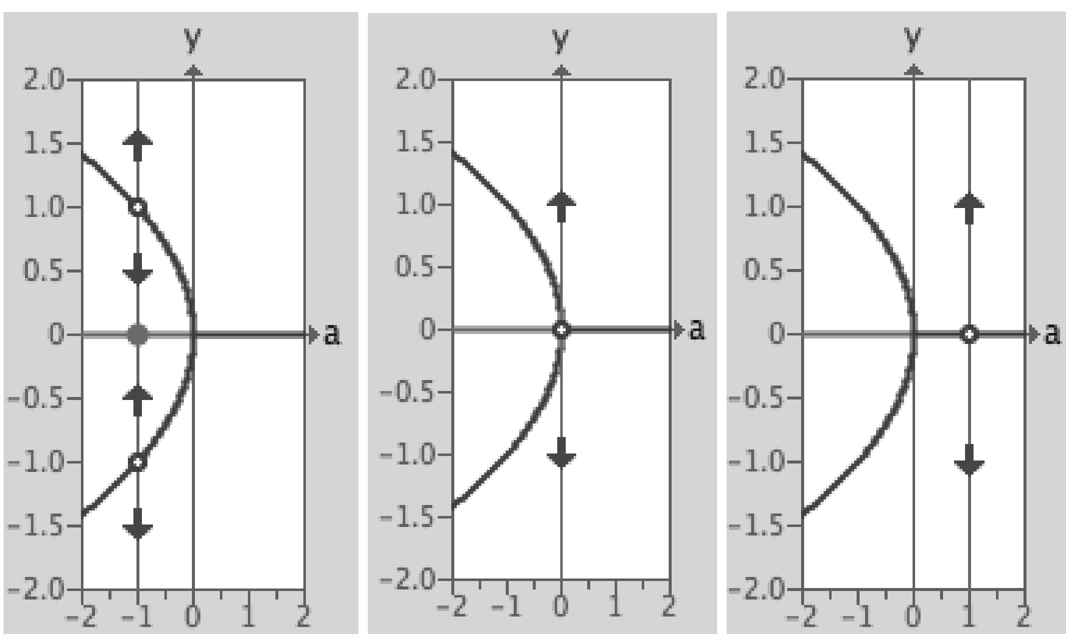
Figure 4.5



$a < 0$ $a = 0$ $a > 0$

By the first derivative test, we see that 0 is a source when $a < 0$ and a sink when $a < 0$. Meanwhile, when $a < 0$, the other 2 equilibria are sources. We see this in the bifurcation diagram below.

Figure 4.6



This type of bifurcation is called a **pitchfork bifurcation**.

Important Terms

bifurcation diagram (bifurcation plane): A picture that contains all possible phase lines for a first-order differential equation, one for each possible value of the parameter on which the differential equation depends. The bifurcation diagram, which plots a changing parameter horizontally and the y value vertically, is similar to a parameter plane, except that a bifurcation diagram includes the dynamical behavior (the phase lines), while a parameter plane does not.

limited population growth model with harvesting: This is the same as the limited population growth model except we now assume that a portion of the population is being harvested. This rate of harvesting can be either constant or periodic in time.

saddle-node bifurcation: In an ODE, this is a bifurcation at which a single equilibrium point suddenly appears and then immediately breaks into two separate equilibria. In a difference equation, a fixed or periodic point undergoes the same change. A saddle-node bifurcation is also referred to as a tangent bifurcation.

pitchfork bifurcation: In this bifurcation, varying a parameter causes a single equilibrium to give birth to two additional equilibrium points, while the equilibrium point itself changes from a source to a sink or from a sink to a source.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 1.7.

Guckenheimer and Holmes, *Nonlinear Oscillations*, chap. 3.1.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 1.3.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 3.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Phase Lines.

Problems

1. Find the equilibrium points of the differential equation $y' = y - A$, and determine their type.

2. Are there any bifurcations in the previous differential equation?

3. Find the equilibrium points of the differential equation $y' = A$, and determine their type.

4. At which A-values does a bifurcation occur in $y' = A$?

5. Find the equilibrium points for $y' = y + Ay^2$.

6. Describe the bifurcation that occurs in the family $y' = Ay$.

7. Plot the bifurcation diagram for $y' = Ay - y^2$.

8. At which type of equilibrium points would you expect a bifurcation to possibly occur in the equation in problem 7?

9. Consider more generally $y' = B + Ay - y^2$. Fix a value of B, and plot the bifurcation diagram that depends only on A. For which values of B do you see a different type of picture?

10. Consider $y' = B + Ay - y^3$. Fix a value of B, and plot the bifurcation diagram that depends only on A. For which values of B do you see a different type of picture?

Exploration

Unlike in our example in this lecture, harvesting is not always constant. For example, harvesting of fish in certain locales is seasonal. Hence it makes more sense to allow the harvesting term to be a periodic function. Toward that end, use a computer to investigate the limited population growth model with periodic harvesting given by

$$y' = y(1 - y) - h(1 + \sin(2\pi t)).$$

Explain the behaviors and bifurcations that you see.

Methods for Finding Explicit Solutions

Lecture 5

In this lecture we describe several standard methods for solving certain types of first-order differential equations, namely, linear and separable first-order equations. To find these explicit solutions, we need to invoke another tool from calculus, integration (or antidifferentiation). Finding an (indefinite) integral or antiderivative is basically the opposite of finding the derivative; the integral of a given function is just a function whose derivative is equal to the given function.

For example, the integral of the function t^n for $n > 0$ is the function $t^{n+1}/(n + 1) + C$, where C is some constant, since the derivative of $t^{n+1}/(n + 1) + C$ is $(n + 1)t^n/(n + 1) = t^n$. We denote the integral of the function $y(t)$ by

$$\int y(t)\,dt\,.$$

The dt indicates that the independent variable is t.

Other examples from calculus that we will use are the following, in which $\ln(t)$ is the natural logarithm function.

$$\int e^{kt}\,dt = e^{kt}/k + C$$

$$\int \frac{dt}{t} = \ln|t| + C$$

$$\int \frac{dt}{1-t} = -\ln|1-t| + C$$

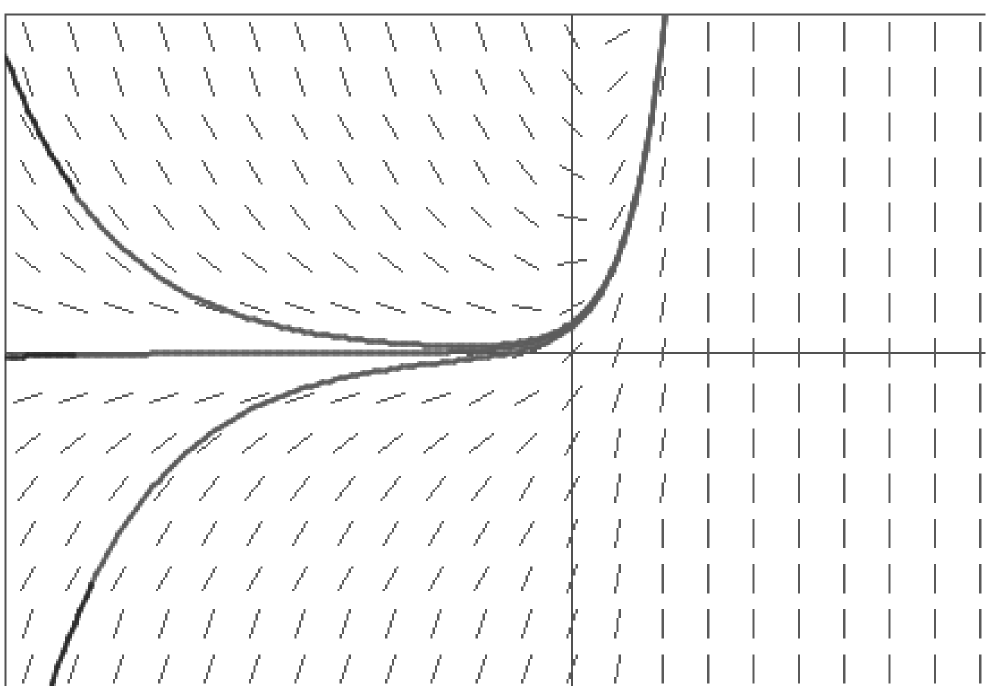
A first-order linear differential equation (with constant coefficients) is a differential equation of the form

$$y' + ky = G(t).$$

The function $G(t)$ may be nonlinear, however, and this expression is sometimes called the forcing term. If $G(t) = 0$ (i.e., we have no forcing term), then our equation is simply $y' = ky$. This is called a first-order linear homogeneous equation, and we know its general solution, $y(t) = Ce^{kt}$. To solve these equations, we use the "guess and check" method. We know how to solve the homogeneous part of this equation; the solution is $y(t) = Ce^{kt}$. We then make an appropriate guess to find the general solution of the nonhomogeneous equation.

As an example of this method, consider $y' + y = e^{3t}$. The solution of the homogeneous equation $y' + y = 0$ is $y(t) = Ce^{-t}$. So we would make a guess of the form $y(t) = Ce^{-t} + Ae^{3t}$. The question is, what is A? Plugging this into the ODE yields $A = 1/4$, so our specific solution is $y(t) = Ce^{-t} + (1/4)e^{3t}$. The solution resembles e^{-t} as time goes backward and e^{3t} as time goes forward.

Figure 5.1



Another example of a first-order linear ODE is **Newton's law of cooling**. Suppose you have a cup of coffee whose temperature at the moment is 170°. Suppose also that the ambient temperature is 70°. Can you find a function $y(t)$ that gives the temperature of your coffee as a function of t? Newton's law of cooling says that the rate of cooling is directly proportional to the difference between the current temperature and the ambient temperature. As an ODE, this says that $y' = k(y - 70)$. If we assume that the coffee is initially cooling at a rate of 20° per minute, our equation at time $t = 0$ reads $-20 = y'(0) = k(y(0) - 70)$. So $k = -0.2$, and the differential equation is $y' = -0.2(y - 70)$ or $y' + 0.2y = 14$, which is first-order, linear, and nonhomogeneous.

We know how to solve the homogeneous equation $y' = -0.2y$; the general solution of this equation is $y(t) = Ce^{-0.2t}$. So plugging this into the left-hand side of the ODE yields 0. We want to find a function that we can plug into the left-hand side to get out the constant 14. So let's guess a solution of the form $y(t) = Ce^{-0.2t} + A$, where A is some constant. Plugging this expression into the differential equation yields $-0.2Ce^{-0.2t} + 0.2Ce^{-0.2t} + 0.2A = 14$, so $A = 70$. Therefore our explicit solution is $Ce^{-0.2t} + 70$. We want the solution that satisfies $y(0) = 170$, so we put this initial value into our solution, which yields $170 = Ce^{0} + 70$, or $C = 100$. Thus our specific solution is $y(t) = 100e^{-0.2t} + 70$.

Another method for solving this equation is **separation of variables**. This equation is separable because we can get all the y's on the left and all the t's on the right. That is, we can rewrite the differential equation given by

$$\frac{dy}{dt} = -0.2(y - 70)$$

in this manner:

$$\frac{dy}{y - 70} = -0.2\,dt\,.$$

Now we integrate both sides of this equation, the left side with respect to y and the right with respect to t:

$$\int \frac{dy}{y-70} = \int -0.2\, dt\ .$$

The integral of the left is just $\ln(y - 70)$ + constant while on the right we find $-0.2t$ + constant. Lumping both of the constants together and calling them D, we find that

$$\ln(y - 70) = -0.2t + D.$$

Exponentiating both sides, we find

$$y - 70 = e^{-0.2t+D} = Ce^{-0.2t},$$

where we have written the term e^D as the constant C. Thus, exactly as above, we find the solution $y(t) = Ce^{-0.2t} + 70$, and this is easily seen to be the general solution.

Now let's return to the limited population growth model given by $y' = y(1 - y)$. This equation is separable, so we are left with doing the 2 integrals:

$$\int \frac{dy}{y(1-y)} = \int dt.$$

The right-hand integral yields t + constant, whereas the left-hand integral is more complicated. We can break up this complicated fraction into 2 "partial fractions" this way:

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y} .$$

So our integrals on the left reduce to

$$\int \frac{dy}{y} + \int \frac{dy}{1-y} = \ln|y| - \ln|1-y| + const .$$

Let's assume that $0 < y < 1$, so both of the terms inside the absolute values are positive. Therefore we are left with

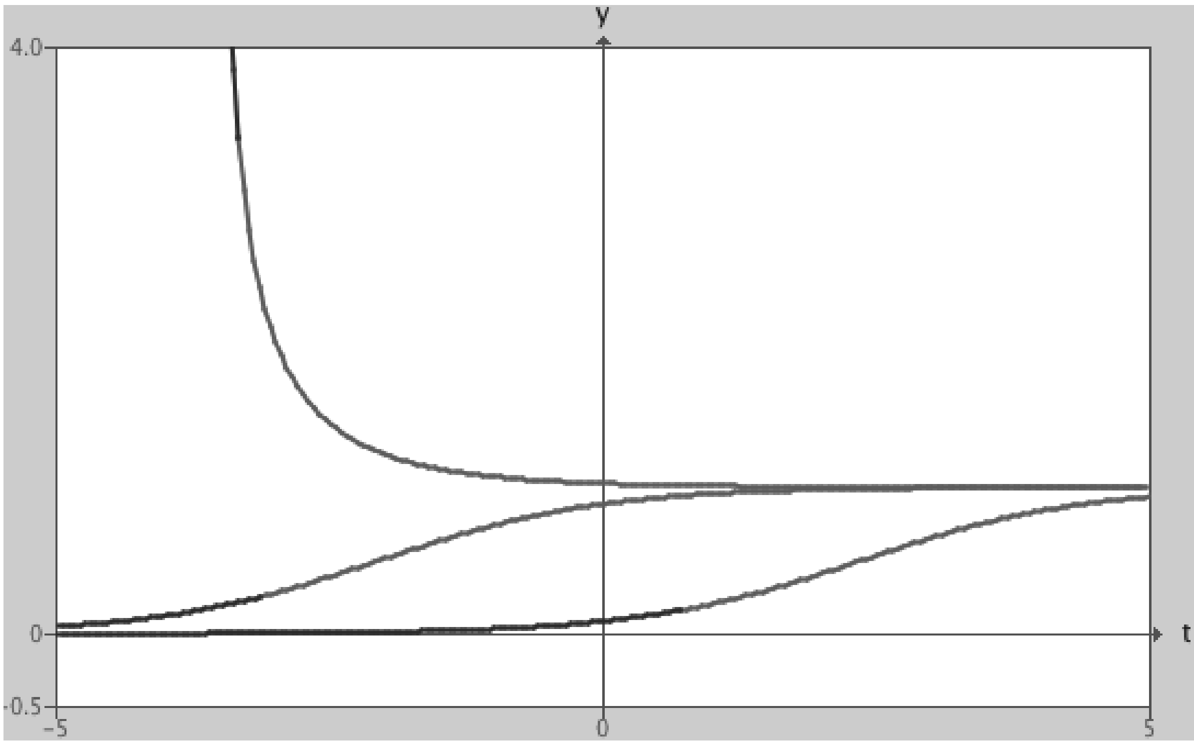
$$\ln(y) - \ln(1-y) = t + C$$

for some constant C. Exponentiation plus some algebra yields

$$y(t) = \frac{De^t}{1+De^t} .$$

These functions have graphs, below, that we have seen before.

Figure 5.2



If we were to assume $y > 1$, we would find the same solution. So is this the general solution? Not quite. If we try to find a value of D that solves the initial value problem $y(0) = y_0$, we must solve the equation

$$y_0 = y(0) = \frac{De^0}{1+De^0} = \frac{D}{1+D}.$$

Doing the algebra yields

$$D = \frac{1}{1 - y_0},$$

so we have found such a D-value as long as y_0 is not equal to 1. But we know the solution to the initial value problem $y(0) = 1$; this solution is just the equilibrium solution $y(t) = 1$. So we do in fact have the general solution once we tack on this solution.

Taking the limit as t goes to infinity of

$$y(t) = \frac{De^t}{1 + De^t} = \frac{D}{e^{-t} + D}$$

yields 1, so all solutions tend to the equilibrium solution $y(t) = 1$, just as we saw earlier.

Important Terms

Newton's law of cooling: This is a first-order ODE that specifies how a heated object cools down over time in an environment where the ambient temperature is constant.

separation of variables: This is a method for finding explicit solutions of certain first-order differential equations, namely those for which the dependent variables (y) and the independent variables (t) may be separated from each other on different sides of the equation.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 1.2.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 1.2.

Roberts, *Ordinary Differential Equations*, chap. 1.3.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 2.3.

Problems

1. First let's review some integral calculus. Find the integrals of the following functions.

a. $y(t) = t + 5$

b. $y(t) = t^2 + 2t + 1$

c. $y(t) = 1/t^2$

d. $y(t) = e^{-t}$

e. $y(t) = 0$

2. Solve the equation $\ln(t) - 4 = 0$.

3. Solve the equation $e^t - 4 = 0$.

4. Find the general solution of $y' - y = 0$, and sketch the graphs of some solutions.

5. Find the general solution of $y' - y = 1$, and sketch the graphs of some solutions.

6. For the unlimited population model with $y > 1$, check that solutions assume the form discussed in the lecture.

7. Find the general solution of the ODE $y' = y^2$.

8. Suppose you have a cup of coffee whose temperature at the moment is 200°, and the ambient temperature is 80°. Suppose that 1 minute later the temperature of the coffee is 180°. Find the function $y(t)$ that gives the temperature of this cup of coffee as a function of t.

9. Find explicit solutions of the differential equation $y' = 1 - y^2$ satisfying the initial conditions $y(0) = 0$, $y(0) = 1$, and $y(0) = 2$.

10. What is the general solution of the equation $y' = 1 - y^2$?

Exploration

If you are familiar with other techniques of integration, you can find solutions of some more complicated differential equations (though most such equations do not have solutions that can be found explicitly). Refresh your knowledge of integration techniques to find the general solutions of the following differential equations.

1. $y' = 1 + y^2$

2. $y' = e^t y/(1 + y^2)$

3. $y' = y + y^3$

How Computers Solve Differential Equations

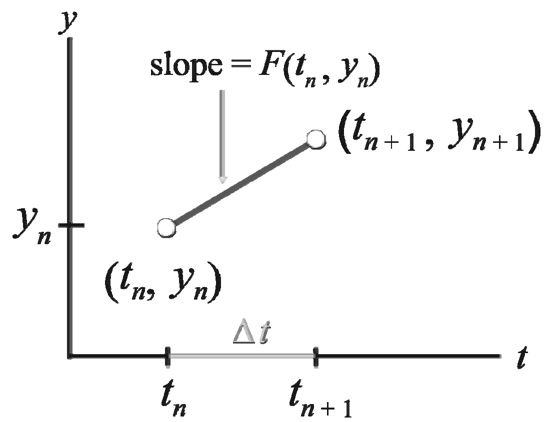
Lecture 6

We have used the computer several times to plot the solutions of differential equations. But how does a computer produce these solutions? Actually, there are many different numerical algorithms for generating these solutions. Perhaps the most widely used method is called Runge-Kutta 4. For simplicity, we'll describe a much easier to understand though less accurate numerical method, **Euler's method**. This algorithm is constructed in much the same way as the more accurate methods: It uses a recursive procedure to approximate the solutions.

Here is the idea behind Euler's method. We wish to approximate the solution to the differential equation $y' = F(y, t)$ starting at the initial value (y_0, t_0). We first choose a step size Δt. This step size will usually be very small. Then we "step" along the slope field, moving by Δt units in the t-direction at each stage. The resulting conglomeration of little straight lines will be the approximation to our solution.

More precisely, starting at the given initial point (y_0, t_0), we will recursively construct a sequence of points (y_n, t_n) for $n = 1, 2, 3, \ldots$ and join each pair of points (y_n, t_n) and (y_{n+1}, t_{n+1}) by a straight line. This straight line is constructed as follows. We start at (y_0, t_0) and draw the slope line out to the point whose t-coordinate is $t_0 + \Delta t$. This is the value of t_1. The corresponding y-value over t_1 is y_1. Then we do the same at our point (y_1, t_1): Draw the slope line and move along it to (y_2, t_2), where $t_2 = t_1 + \Delta t$.

Figure 6.1



We therefore have recursively $t_{n+1} = t_n + \Delta t$. So the only question is how do we generate the value of y_{n+1} knowing both y_n and t_n? The answer comes from algebra. We have the straight line segment passing through the known point (y_n, t_n) and having slope given by the right-hand side of the ODE—that is to say, the slope is $F(y_n, t_n)$, whose value we know. Therefore our little slope field line has equation

$$y = Mt + B,$$

where $M = F(y_n, t_n)$ is the slope and B is the y-intercept. To determine the value of B, we know that the point (y_n, t_n) lies on our line. So we have

$$y_n = F(y_n, t_n)t_n + B.$$

Therefore

$$B = y_n - F(y_n, t_n)t_n.$$

So the equation for y_{n+1} reads

$$\begin{aligned} y_{n+1} &= F\left(y_n, t_n\right)(t_n + \Delta t) + y_n - F\left(y_n, t_n\right)t_n \\ &= y_n + F\left(y_n, t_n\right)\Delta t. \end{aligned}$$

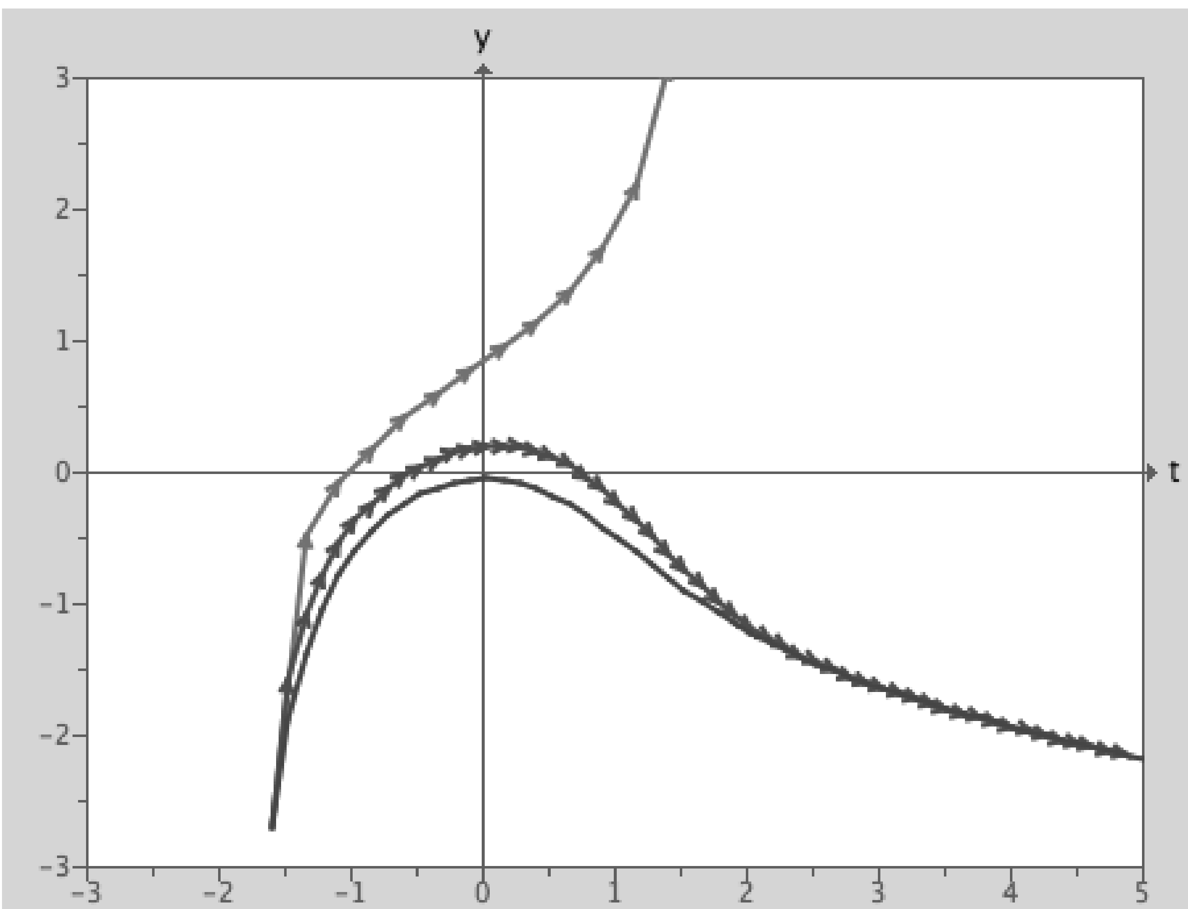
Thus the recursive formula to generate the values y_{n+1} and t_{n+1} is given by

$$t_{n+1} = t_n + \Delta t$$

$$y_{n+1} = y_n + F(y_n, t_n)\,\Delta t\ .$$

Below is an image of Euler's method applied to the differential equation $y' = y^2 - t$ with inordinately large step sizes of 0.25 and 0.125. Also displayed is the actual solution starting at (y_0, t_0). The arrows are little segments of the slope field. Note how when the step size is 0.25, our approximate solution is way off, but when we lower the step size to 0.125, we get a better approximation. Lowering the step size further and further usually gives better and better approximations.

Figure 6.2



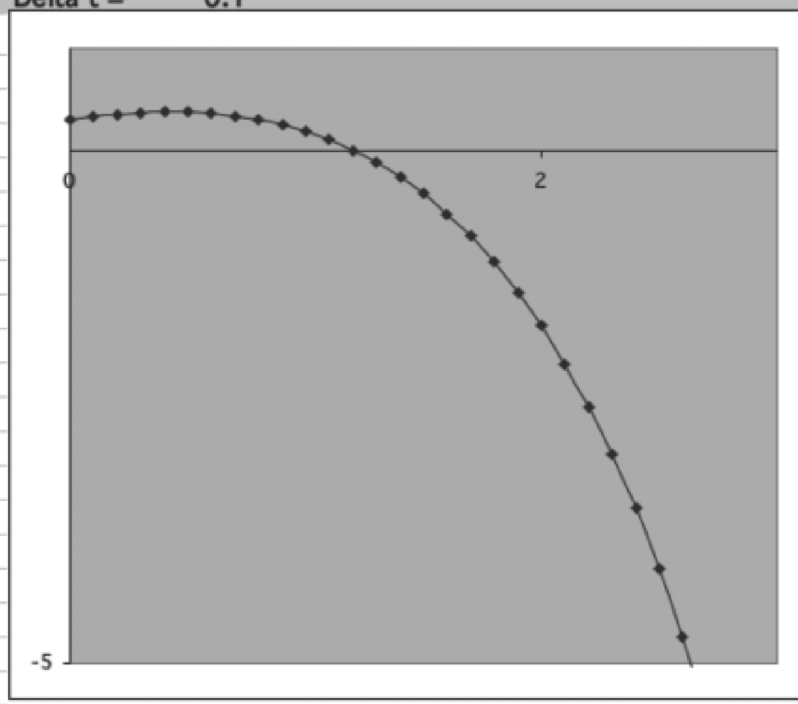
Perhaps the easiest way nowadays to invoke Euler's method is to use a spreadsheet. Here, for example, is a spreadsheet calculation of the solution to the differential equation $y' = y - t$ starting from the initial position $y(0) = 0.3$.

Figure 6.3

y'= y - t

Delta t = 0.1

n	t_n	y_n
0	0	0.3
1	0.1	0.33
2	0.2	0.353
3	0.3	0.3683
4	0.4	0.37513
5	0.5	0.372643
6	0.6	0.3599073
7	0.7	0.33589803
8	0.8	0.29948783
9	0.9	0.24943662
10	1	0.18438028
11	1.1	0.10281831
12	1.2	0.00310014
13	1.3	-0.1165899
14	1.4	-0.2582488
15	1.5	-0.4240737
16	1.6	-0.6164811
17	1.7	-0.8381292
18	1.8	-1.0919421
19	1.9	-1.3811363
20	2	-1.70925
21	2.1	-2.080175
22	2.2	-2.4981925
23	2.3	-2.9680117



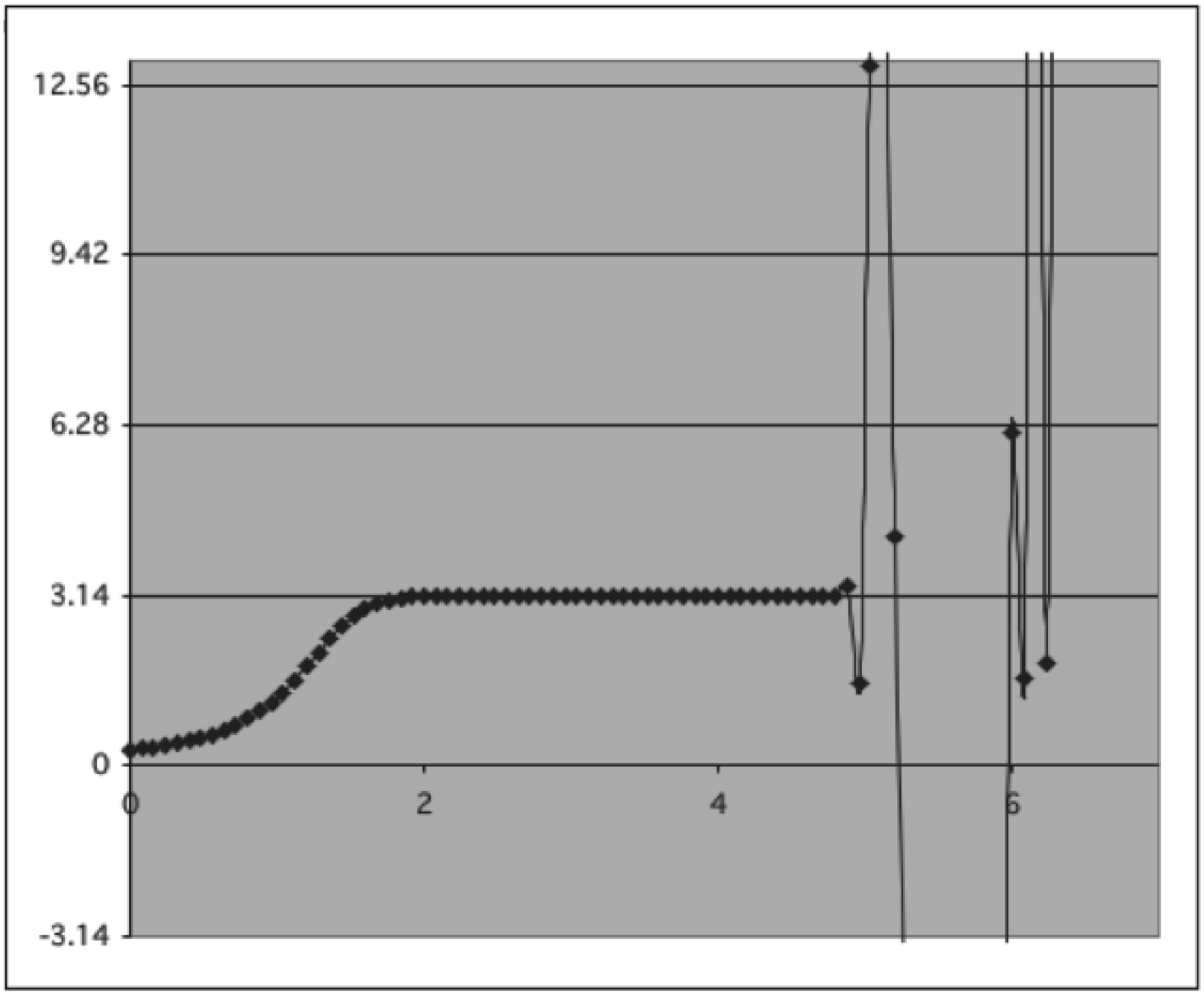
y(1) =	0.09720272
Euler:	0.18438028
Error:	0.08717756

Above we have used the step size $\Delta t = 0.1$ and displayed the corresponding sequence of straight line segments.

Unfortunately, no numerical algorithm is flawless; we can always find a differential equation that breaks a given numerical method. For example, consider the differential equation $y' = e^t \sin(y)$. Clearly there are equilibrium

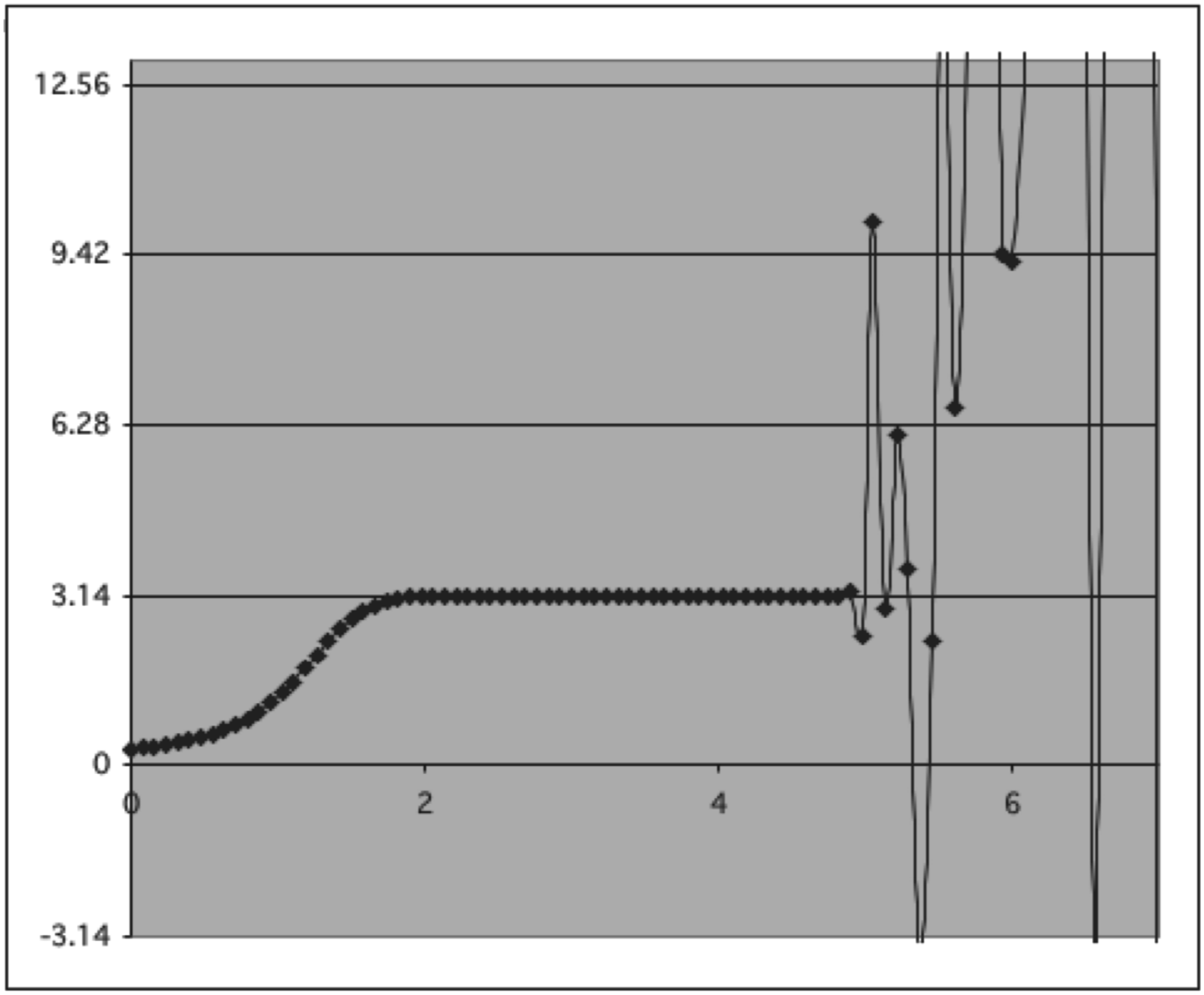
points at $y = 0, \pm\pi, \pm 2\pi, \ldots$. So a solution that starts at $y(0) = 0.3$ should simply increase up to $y = \pi$ as time moves on. But you can see below a time series for what Euler's method yields with a step size of .08. Clearly, the results are bad. What you are actually seeing is chaotic behavior, another theme we will return to later.

Figure 6.4



If we change the step size just a little bit to .079, we get very different results for Euler's method. This is sensitive dependence on initial conditions, the hallmark of the phenomenon known as chaos.

Figure 6.5



Important Term

Euler's method: This is a recursive procedure to generate an approximation of a solution of a differential equation. In the first-order case, basically this method involves stepping along small pieces of the slope field to generate a "piecewise linear" approximation to the actual solution.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 1.4 and 7.1–7.2.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 7.5.

Roberts, *Ordinary Differential Equations*, chap 2.4.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 2.8.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Euler's method.

Problems

1. First let's try some refreshers from algebra in the t-y plane.

a. What is the slope of the straight line in the plane connecting (1, 1) to (3, 4)?

b. What is the slope of the straight line connecting (t_0, y_0) to (t_1, y_1)?

c. What is the equation of the straight line connecting (1, 0) to (0, 1)?

d. What is the equation of the straight line connecting (1, 0) to (3, 0)?

e. What is the equation of the straight line connecting (1, 0) to (1, 3)?

2. What is the equation of the tangent line to the graph of $y = t^2$ at the point $t = 1$?

3. Consider the differential equation $y' = y$. Given the initial condition $t_0 = 0$, $y_0 = 1$ and step size 0.1, use the formula for Euler's method to compute by hand t_1 and y_1.

4. Repeat the above to compute t_2 and y_2 as well as t_3 and y_3.

5. Sketch what you have done via Euler's method for the approximate solution so far.

6. **a.** For the differential equation $y' = y$, find the solution satisfying $y(0) = 1$.

   b. Then use Euler's method with a step size of 0.1 to approximate the value of $y(1)$.

   c. Then recompute using step sizes of 0.05 and 0.01. How do these compare to the approximation above with step size 0.1?

   d. Using the value of e as approximately 2.718281, how does the error change as we vary the step size?

7. Use Euler's method to approximate the solution of the differential equation $y' = (1 + y)/(1 - y)$ with any initial condition $y(0) \neq 1$. What happens here?

Exploration

A slightly better numerical approximation is given by the so-called improved Euler's method. This method also uses a step size Δt. However, instead of using the slope field at the point (y_n, t_n) to approximate the solution, this method uses a line with a slightly different slope. To get this slope, first use the Euler's method approximation at the point (y_n, t_n) to find the endpoint of the slope line at this point whose t-coordinate is $t_n + \Delta t$. Then take the average of the slopes at these 2 points.

This is the slope of the straight line for the improved Euler's method. Find the formula for this slope line. Now use the improved Euler's method to solve the above differential equation with step sizes 0.1, 0.05, and 0.01. How does the improved Euler's method compare to the ordinary Euler's method?

Systems of Equations—A Predator-Prey System

Lecture 7

We now begin the main part of this course, which deals with systems of differential equations. Our first example is the predator-prey system. Here we assume we have 2 species living in a certain environment. One species is the prey population (we'll use rabbits), and the other species is the predator population (we'll use foxes). Denote the prey population by $R(t)$ and the predator population by $F(t)$.

We assume first that if there are no foxes around, the rabbit population obeys the unlimited population growth model. If, however, foxes are present, then we assume that the rate of growth of the rabbit population decreases at a rate proportional to the number of rabbit and fox encounters, which we can measure by the quantity RF. So the differential equation for the rabbit population can be written

$$\frac{dR}{dt} = aR - bRF ,$$

where a and b are parameters.

For the fox population, our assumptions are essentially the opposite; so the differential equation for $F(t)$ is

$$\frac{dF}{dt} = -cF + dRF .$$

Again, c and d are parameters.

Note that both of these equations depend on $R(t)$ and $F(t)$. This is typical of systems of differential equations: We have a collection of differential equations involving a number of dependent variables, and each equation

depends on the other dependent variables. So the **predator-prey system** is given by the following equations.

$$\frac{dR}{dt} = aR - bRF$$

$$\frac{dF}{dt} = -cF + dRF$$

This is a 2-dimensional system of ODEs, since we have just 2 dependent variables.

More generally, a system of ODEs is a collection of n first-order differential equations for the missing functions $y_1, \ldots, y_n$. Each equation depends on all of the dependent variables $y_1, \ldots, y_n$ as well as (possibly) t. So we get the following.

$$\begin{aligned} \frac{dy_1}{dt} &= F_1(y_1, \ldots, y_n, t) \\ &\vdots \\ \frac{dy_n}{dt} &= F_n(y_1, \ldots, y_n, t) \end{aligned}$$

This is an n-dimensional system of differential equations. Later we will spend a lot of time looking at linear systems of equations with constant coefficients. A 2-dimensional version of such a system is

$$x' = ax + by$$

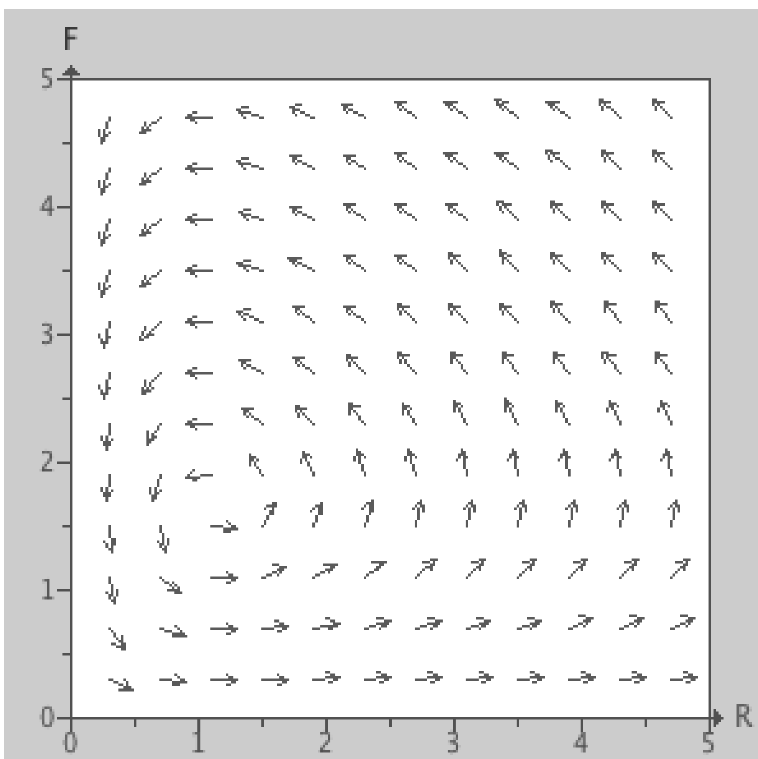
$$y' = cx + dy$$

where a, b, c, and d are constants.

A solution of this system of ODEs is then a pair of functions, $(R(t), F(t))$, each of which depends on the independent variable. We will think of this pair of functions as a curve in the plane parameterized by t. We call this curve a **solution curve**. Thus the plane becomes our **phase plane**, in analogy with the phase line viewed earlier.

Note that we know the tangent vector to any given solution curve: This is the vector given by the right-hand side of our pair of differential equations (R', F'). So the right-hand side of our equation determines a **vector field** in the phase plane. Solution curves in the R-F plane are everywhere tangent to the vector field. Since these vectors can often be very large (and therefore cross each other in ways that make viewing the vector field difficult), we usually scale the vector field so that all vectors have the same length; this scaled field is called the **direction field**.

Figure 7.1



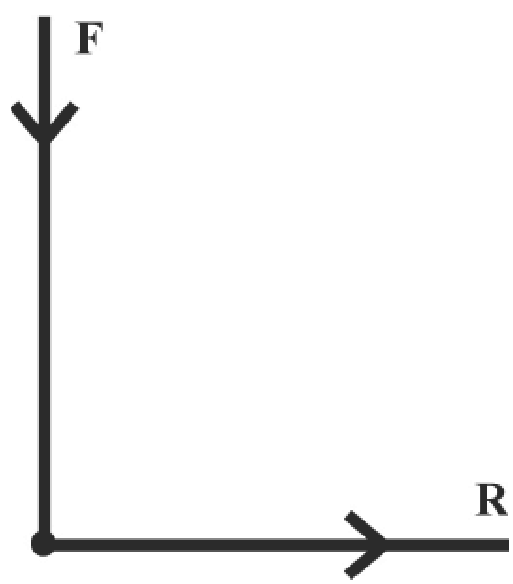
The point (R_0, F_0) is an equilibrium point if the right-hand sides of both differential equations vanish at this point. This means that we have a constant solution for the system given by $R(t) = R_0$ and $F(t) = F_0$. We indicate this by a dot in the phase plane. So the equilibrium points for the predator-prey system are given by solving the pair of algebraic equations simultaneously:

$$R' = aR - bRF = 0, F' = -cF + dRF = 0.$$

Solving this shows that there are equilibria at (0, 0) and $(c/d, a/b)$.

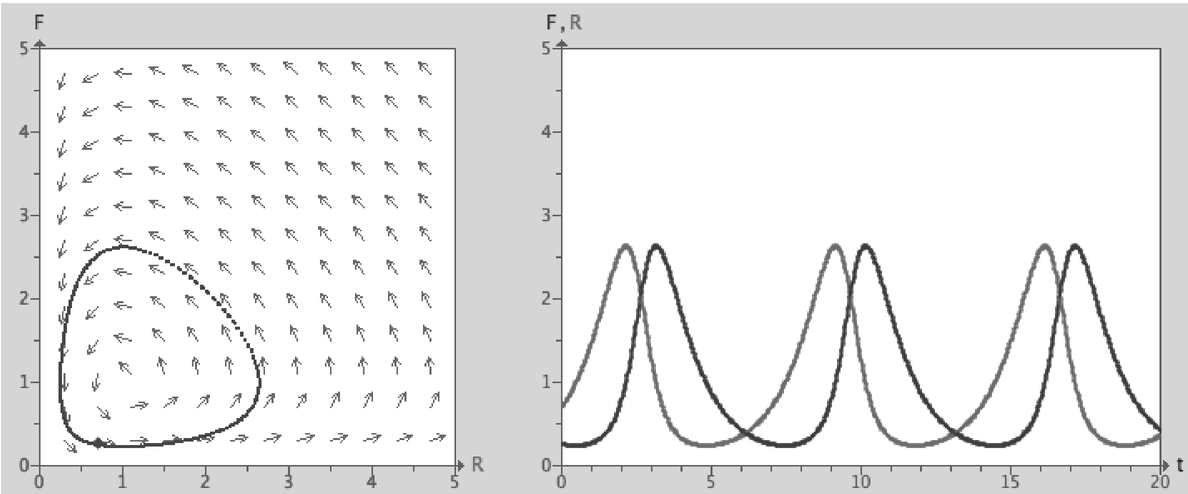
Note that when $R = 0$, we have $R' = 0$, so the rabbit population stays fixed at 0. Also, in this case, the equation for foxes becomes $F' = -cF$. This is just our old population model, this time with population decline rather than growth. So, along the F-axis, we can see the phase line corresponding to this first-order equation. Similarly, when there are no foxes, the differential equation for rabbits becomes $R' = aR$, the unlimited population growth model for rabbits. Thus we see the phase line for this equation along the R-axis in the phase plane.

Figure 7.2



We may then sketch the vector field in the special case where $a = b = c = d = 1$ and use numerical methods to superimpose the plots of various solution curves.

Figure 7.3



On the left, we see a solution in the phase plane. Here the R-axis is the horizontal axis and the F-axis is the vertical axis. This solution winds repeatedly around the nonzero equilibrium point. On the right, we have superimposed the corresponding graphs of both $R(t)$ and $F(t)$. Now the t-axis is the horizontal axis, while the F- and R-axes are vertical.

We may modify the predator-prey system by assuming that the rabbit population obeys the limited growth model with carrying capacity $N = 1$. For simplicity we'll choose all the other parameters to be equal to 1, except d. So our new system is

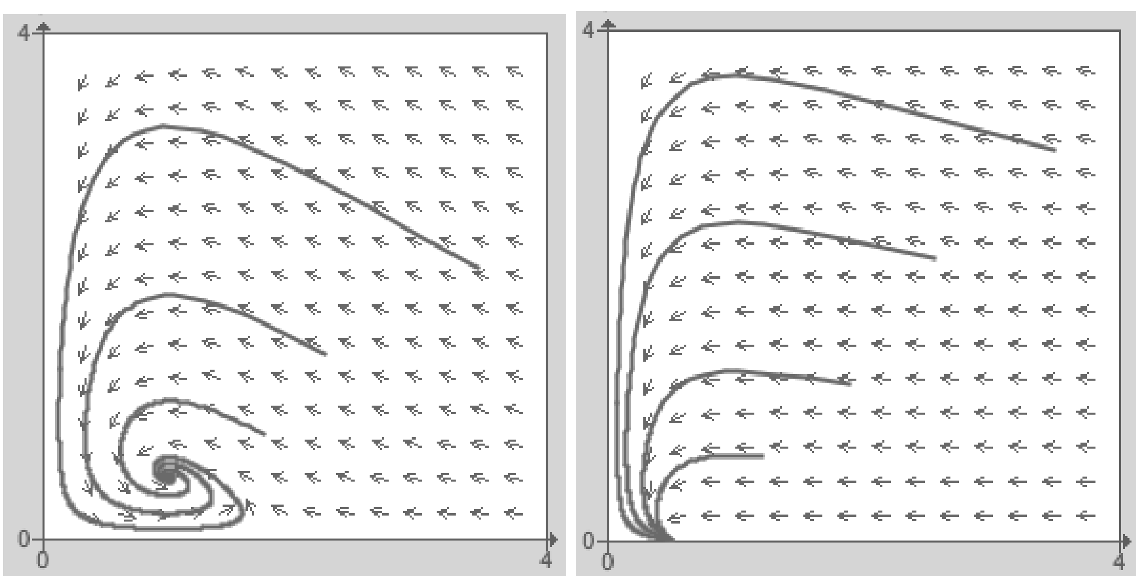
$$\frac{dR}{dt} = R(1-R) - RF$$

$$\frac{dF}{dt} = -F + d\,RF\ .$$

On the R-axis, our phase line is now the limited growth phase line with equilibrium points at $R = 0$ (a source) and $R = 1$ (a sink). On the F-axis, we have (as before) the fox population tending to the equilibrium at 0. Setting the 2 differential equations equal to zero yields 2 equilibrium points, at the origin and (1, 0). But if $d > 1$, we get a new equilibrium point at $(R = 1/d, F = (d - 1)/d)$. If $0 < d < 1$, there is an equilibrium point at $(R = 1/d, F = (d - 1)/d)$, but the F-coordinate here is negative, so we will not consider this in our population model. Thus we have a bifurcation when $d = 1$.

But there is more to this story. When $d \leq 1$, the phase plane appears to show that any solution that begins with a positive fox population now tends to the equilibrium point at (1, 0). That is, the fox population goes extinct. But when $d > 1$, the solutions seem to tend to the equilibrium point $(1/d, (d - 1)/d)$, so both populations seem to stabilize as time goes on. This is another example of a bifurcation.

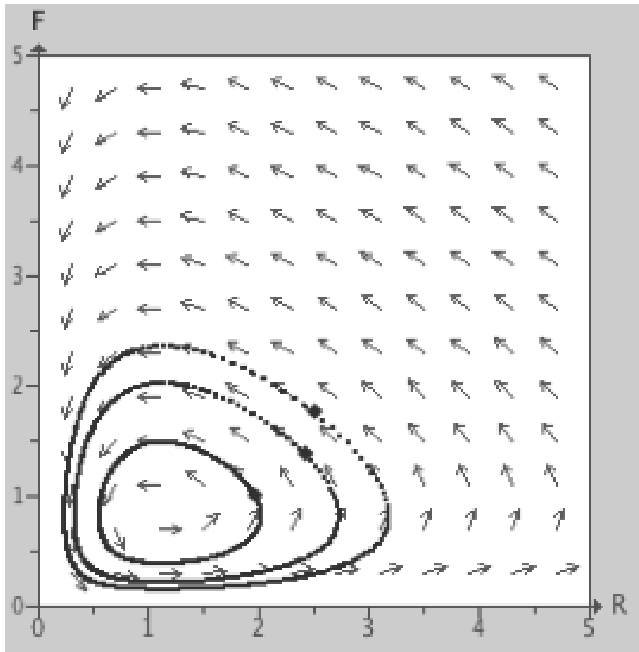
Figure 7.4



Phase plane for $d = 2$ Phase plane for $d = 1/2$

Many new types of solutions appear in systems of ODEs. For example, there may be periodic solutions as in the original predator-prey system.

Figure 7.5



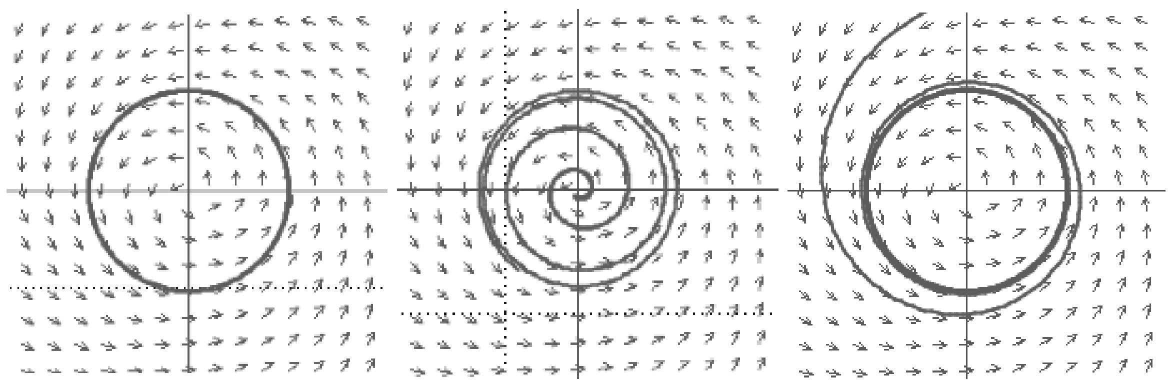
And solutions may spiral toward or away from such a periodic solution. For example, consider the following system of equations.

$$x' = (1 - \sqrt{x^2 + y^2})x - y$$
$$y' = x + (1 - \sqrt{x^2 + y^2})y$$

It is very rare to be able to find periodic solutions explicitly for systems of ODEs, but here we can. Note that when $x^2 + y^2 = 1$, the system reduces to $x' = -y$ and $y' = x$. Do you know a pair of functions that satisfies this relation? Sure—from trigonometry we know that $\cos^2(t) + \sin^2(t) = 1$, and from calculus we know that the derivative of $\cos(t)$ is $-\sin(t)$, while the derivative of $\sin(t)$ is $\cos(t)$. So $x(t) = \cos(t)$, $y(t) = \sin(t)$ is a solution of this system, which is the periodic solution we see above.

Solutions spiral in toward a periodic solution (below).

Figure 7.6



Another type of differential equation that comes up often is a second-order equation. This is an equation that involves both y' and y''. We will deal for the most part with second-order equations of the form

$$Y'' + ay' + by = F(t),$$

where $F(t)$ is considered a forcing term. For example, in the next lecture we will consider the mass-spring system given by $y'' + by' + ky = 0$. We can write this equation as a system by introducing the new variable $v = y'$ (the velocity). Then our system becomes

$$y' = v$$

$$v' = -ky - bv.$$

Note that this is a linear system of ODEs. This will enable us to see solutions both as graphs of $y(t)$ and also as curves $(y(t), v(t))$ in the phase plane.

Important Terms

direction field: This is the vector field each of whose vectors is scaled down to be a given (small) length. We use the direction field instead of the vector field because the vectors in the vector field are often large and overlap each other, making the corresponding solutions difficult to visualize. Those solutions and the direction field appear within the phase plane.

phase plane: A picture in the plane of a collection of solutions of a system of two first-order differential equations: $x' = F(x, y)$ and $y' = G(x, y)$. Here each solution is a parametrized curve, $(x(t), y(t))$ or $(y(t), v(t))$. The term "phase plane" is a holdover from earlier times when the state variables were referred to as phase variables.

predator-prey system: This is a pair of differential equations that models the population growth and decline of a pair of species, one of whom is the predator, whose population only survives if the population of the other species, the prey, is sufficiently large.

solution curve (or graph): A graphical representation of a solution to the differential equation. This could be a graph of a function $y(t)$ or a parametrized curve in the plane of the form $(x(t), y(t))$.

vector field: A collection of vectors in the plane (or higher dimensions) given by the right-hand side of the system of differential equations. Any solution curve for the system has tangent vectors that are given by the vector field. These tangent vectors (and, even more so, the scaled-down vectors of a corresponding direction field) are the higher-dimensional analogue of slope lines in a slope field.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 2.1.

Edelstein-Keshet, *Mathematical Models in Biology*, chap. 6.2.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 11.2.

Roberts, *Ordinary Differential Equations*, chap. 10.5.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Predator Prey.

Problems

1. Find all of the equilibrium points for the system below.

$$x' = x + y$$

$$y' = y$$

2. Find the equilibrium points for the system below.

$$x' = x + y$$

$$y' = x + y$$

3. **a.** Sketch the direction field for the system below.

$$x' = x$$

$$y' = y$$

b. What happens to solutions of this system?

c. Find a formula for the solutions of this system.

4. Write the second-order differential equation $y'' = y$ as a system of differential equations.

5. **a.** Sketch the direction field for the following system of differential equations.

$$x' = y$$

$$y' = -x$$

b. Can you find some pairs of solutions $(x(t), y(t))$ of this system of differential equations?

6. **a.** Sketch the direction field for the following system of differential equations.

$$x' = x(1 - x)$$

$$y' = -y$$

b. What are the equilibrium points for this system?

c. This system decouples in the sense that the equation for x' does not depend upon y, and the equation for x' does not depend on x. Use this fact to describe the behavior of all solutions in the phase plane.

Exploration

It is impossible to find explicit solutions of the predator-prey equations described in this lecture. However, there is a technique from multivariable calculus that can give explicit formulas for where solutions lie. That is, we can find a real valued function $H(x, y)$ that is constant along each solution. So plotting the level curves of this function shows us the layout of the solutions. Your job is to use calculus to find such a function. Hint 1: Find a function H such that dH/dt is identically equal to zero. This will involve the chain rule. Hint 2: A function of the form $H(x, y) = F(x) + G(y)$ works.

Second-Order Equations—The Mass-Spring System

Lecture 8

In this lecture, we introduce a slightly different kind of differential equation, second-order (autonomous) linear differential equations. These are differential equations of the form

$$y'' + ay' + by = G(t).$$

As usual, we begin with a model, this time the **mass-spring system**, also known as the harmonic oscillator. Imagine holding a spring hanging with a weight (the mass) attached. If you pull the mass downward (or push it straight upward) and let it go, the mass will then move along a vertical line. Let the position of the mass be $y = y(t)$. The place where the mass is at rest would be called $y = 0$, whereas $y < 0$ if the spring is stretched and $y > 0$ if the spring is compressed.

To write the differential equation for the motion of the mass, we invoke Newton's law that says that the force acting on the mass is equal to the mass times its acceleration. There are 2 types of forces acting on the mass: One is the force exerted by the spring itself, and the second is the force arising from friction (like air resistance). For the force exerted by the spring, we invoke Hooke's law that says that the force exerted by the spring is proportional to the spring's displacement from its rest position and is exerted toward the rest position. So this force is $-ky$. Here the constant $k > 0$ is the **spring constant**.

For the force arising from friction, we make the simple assumption that the force is proportional to the velocity. So this damping force is given by $-by'$, where b is the **damping constant**. Here we either have $b = 0$ (the undamped case) or $b > 0$ (the damped case). The minus sign here indicates that the

damping pushes against the direction of the motion, thereby reducing the speed. This gives us the equation for the damped harmonic oscillator:

$$my'' = -by' - ky.$$

To simplify matters, we usually assume that the mass equals 1. So the equation for the damped harmonic oscillator is

$$y'' + by' + ky = 0.$$

Usually such a second-order differential equation comes with a pair of initial conditions: the initial position of the spring $y(0)$ and its initial velocity $y'(0)$.

We can write the equation for the harmonic oscillator as a system of differential equations by introducing a new variable, the velocity $v = y'$. So the system of equations is

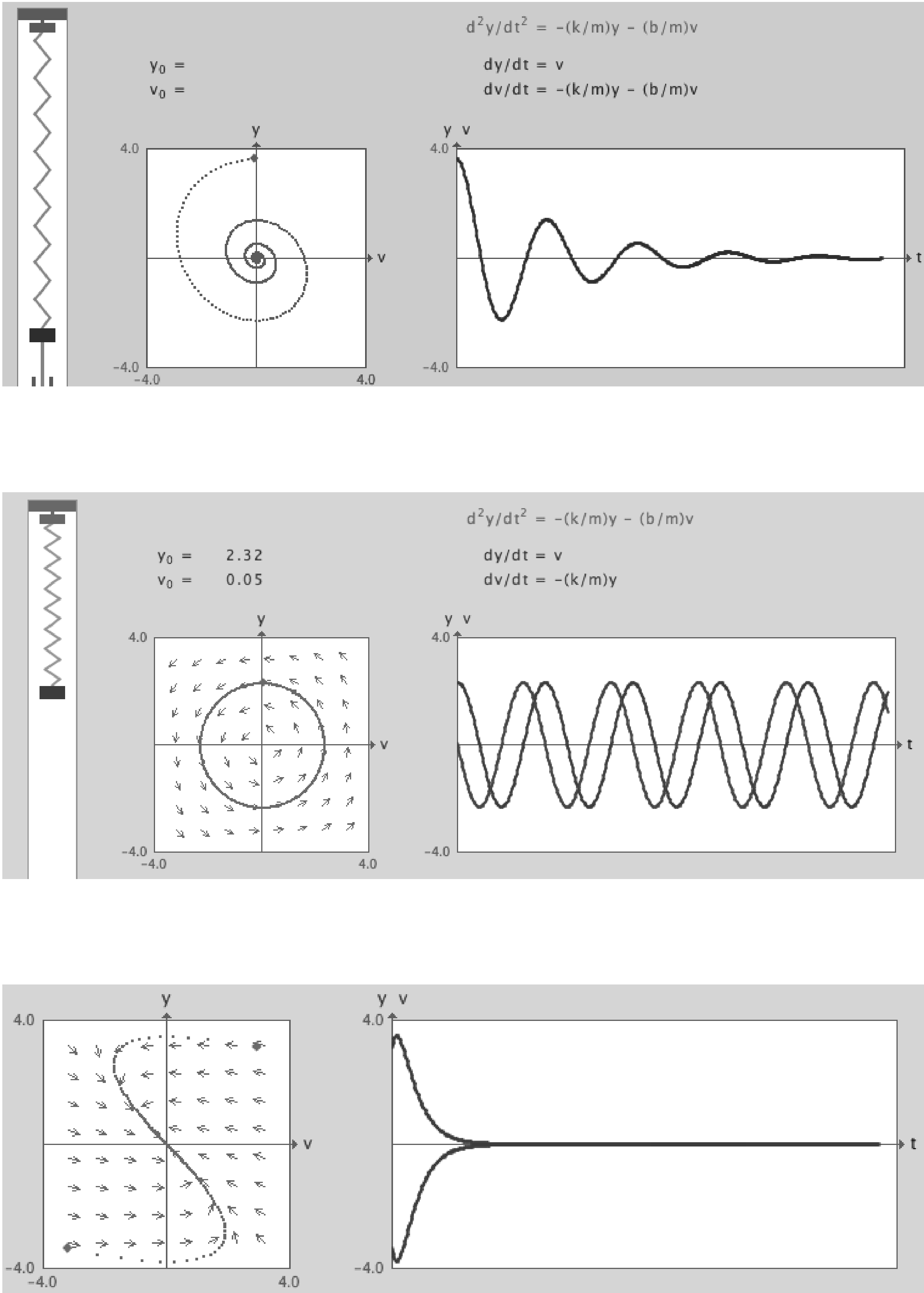
$$y' = v$$

$$v' = -ky - bv,$$

where $k > 0$ and $b \geq 0$. Since $k > 0$, we see that the only equilibrium point for this system is at the origin (i.e., where the mass attached to the spring is at rest). We now have 3 different views of the mass-spring system: the actual motion of the mass on the left, the solution in the phase plane in the center, and the graphs of both position $y(t)$ and velocity $v(t)$ on the right.

We also get 3 different types of solutions: one where the mass oscillates back and forth as it goes to rest, one where it oscillates forever, and one where it proceeds directly to the rest position.

Figure 8.1



$d^2y/dt^2 = -(k/m)y - (b/m)v$
$y_0 =$
$v_0 =$
$dy/dt = v$
$dv/dt = -(k/m)y - (b/m)v$
$d^2y/dt^2 = -(k/m)y - (b/m)v$
$y_0 =$ 2.32
$v_0 =$ 0.05
$dy/dt = v$
$dv/dt = -(k/m)y$

How do we solve such a second-order linear equation? Recall that, for the analogous first-order equation $y' = ky$, we always had the solution $y(t) = Ce^{kt}$. So why not guess a solution of the form e^{st} and try to determine what value of s works? Plug e^{st} into the equation $y'' + by' + ky = 0$. We find

$$e^{st}(s^2 + bs + k) = 0.$$

Therefore any root of the equation $s^2 + bs + k = 0$ yields an exponential function that solves the equation. This quadratic equation is called the **characteristic equation**; we will see many other cases where this equation arises. By the quadratic formula, we find that the roots are

$$\frac{-b \pm \sqrt{b^2 - 4k}}{2}.$$

But there is a problem here. What happens if the quantity $b^2 - 4k$ is negative? Then the roots of our characteristic equation are complex numbers. What is the exponential of a complex number? We'll tackle that problem a little later. Notice also that, if $b^2 - 4k = 0$, then we get only 1 root of the characteristic equation, namely $-b/2$. So we get 1 exponential solution $e^{-bt/2}$. This will not give enough solutions to solve every initial value problem, so more must be done here as well. We'll come back to this situation again later.

Let's suppose that $b^2 - 4k > 0$. Then we have 2 distinct and real roots of the characteristic equation. Let's call them α and β, where the following are true.

$$\alpha = (-b - \sqrt{b^2 - 4k}) / 2$$

$$\beta = (-b + \sqrt{b^2 - 4k}) / 2$$

Clearly, $\alpha < 0$. We also have $\beta < 0$ since $b^2 - 4k < b^2$ (remember that k, our spring constant, is always positive). So we have 2 decreasing exponential solutions, $\exp(\alpha t)$ and $\exp(\beta t)$. And you can check that the general solution is

$$k_1 e^{\alpha t} + k_2 e^{\beta t}.$$

Anytime we find 2 such real, distinct, and negative roots of the characteristic equation, our harmonic oscillator is overdamped. This means that the mass slides down to rest without oscillating back and forth.

As an example, suppose the spring constant is 2 and the damping constant is 3. So the equation for this mass-spring system is

$$y'' + 3y' + 2y = 0.$$

The characteristic equation is

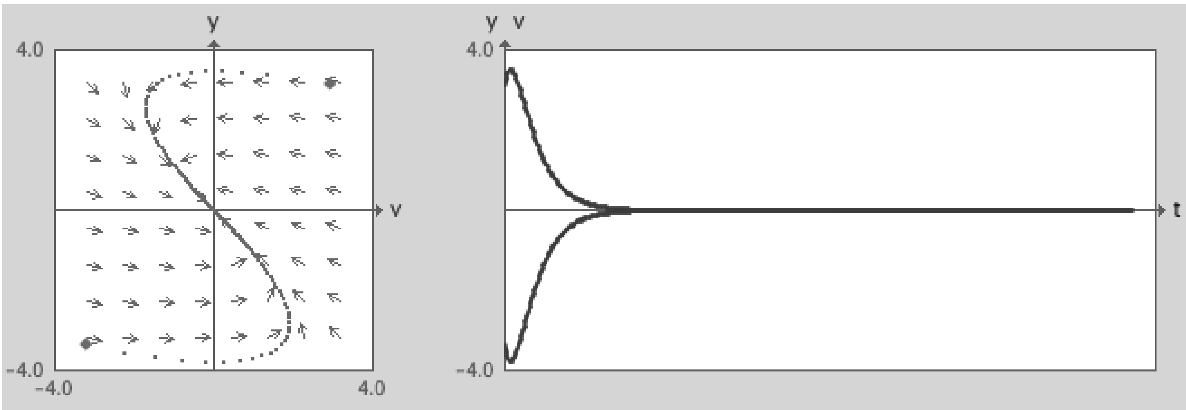
$$s^2 + 3s + 2 = 0,$$

whose roots are -1 and -2. So the general solution here is

$$y(t) = k_1 e^{-t} + k_2 e^{-2t}.$$

Here are the phase plane and the $y(t)$ plots for several of these solutions.

Figure 8.2



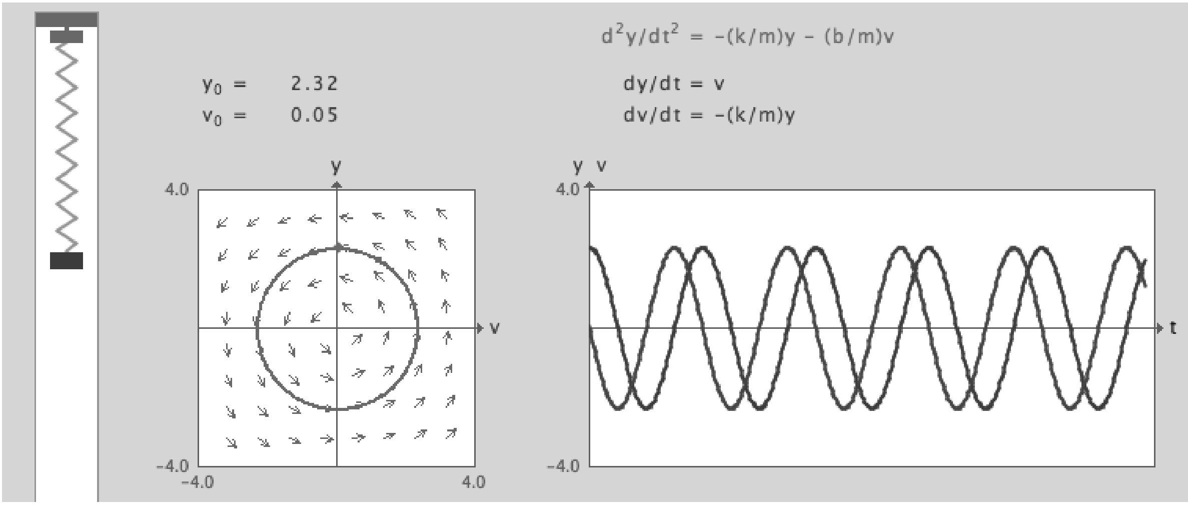
Note that both of these solutions tend directly to the equilibrium point at the origin. The corresponding motion of the mass-spring system is not the usual oscillation back and forth, eventually coming to rest. It's as if we were playing with this mass-spring system in an environment with a lot of resistance, like under water. This is the behavior of the overdamped harmonic oscillator.

So let's now turn to the undamped harmonic oscillator. So $b = 0$, and the equation is $y'' + ky = 0$ or $y'' = -ky$. Do you know a function for which the second derivative is $-k$ times the function? What if $k = 1$? Do you know a function for which $y'' = -y$? Of course, both $\sin(t)$ and $\cos(t)$ have this property. For other k-values, $\sin(k^{1/2}t)$ and $\cos(k^{1/2}t)$ have this property. And we can easily check that the general solution is of the form

$$c_1 \sin(k^{1/2}t) + c_2 \cos(k^{1/2}t).$$

That's why we see the periodic behavior of solutions; any such function is periodic with period $2\pi/k^{1/2}$.

Figure 8.3



Important Terms

characteristic equation: A polynomial equation (linear, quadratic, cubic, etc.) whose roots specify the kinds of exponential solutions (all of the form e^{At}) that arise for a given linear differential equation. For a second-order linear differential equation (which can be written as a two-dimensional linear system of differential equations), the characteristic equation is a quadratic equation of the form $\lambda^2 - T\lambda + D$, where T is the trace of the matrix and D is the determinant of that matrix.

damping constant: A parameter that measures the air resistance (or fluid resistance) that affects the behavior of the mass-spring system. Contrasts with the spring constant.

mass-spring system: Hang a spring on a nail and attach a mass to it. Push or pull the spring and let it go. The mass-spring system is a differential equation whose solution specifies the motion of the mass as time goes on. The differential equation depends on two parameters, the spring constant and the damping constant. A mass-spring system is also called a harmonic oscillator.

spring constant: A parameter in the mass-spring system that measures how strongly the spring pulls the mass. Contrasts with the damping constant.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 2.1, 3.6.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 6.2.

Roberts, *Ordinary Differential Equations*, chap. 6.1.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 5.1.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Mass Spring.

Problems

1. Write the second-order differential equation for the mass-spring system with mass = 1, spring constant = 2, and damping constant = 3.

2. Convert this second-order equation to a system.

3. Find all equilibrium points for this system.

4. Compute the second derivative of the functions $\sin(2t)$ and $\cos(2t)$.

5. Sketch the curve (e^{-t}, e^{-t}) in the plane.

6. For the differential equation $y'' + 3y' + 2y = 0$, show that $y(t) = k_1e^{-t} + k_2e^{-2t}$ is the general solution. Do this by showing that you can generate a solution of this form that solves any initial value problem $y(0) = A$, $y'(0) = B$.

7. **a.** Find the general solution of the mass-spring system with spring constant 6 and damping constant 5.

   b. Find the solution of the above mass-spring system that starts at $y(0) = 0$ with $y'(0) = 1$.

   c. Sketch both the graph of $y(t)$ and the corresponding curve $(y(t), v(t))$ in the phase plane.

   d. Explain in words what actually happens to this mass as time unfolds.

Exploration

Later in the course, we will investigate what happens to the mass-spring system when we apply an external periodic forcing. As a prelude to this, investigate what happens when we apply other, easier forcing terms. Consider first the mass-spring system in problem 2 above, with spring constant 6 and damping constant 5. What happens if we apply a constant force, say 1, to this system? That is, what happens to solutions of the ODE $y'' + 5y' + 6y = 1$? How about the systems where the forcing is 1 from time 0 to time 5, and then we turn the forcing off? Find the solution to this equation that satisfies $y(0) = y'(0) = 0$.

Damped and Undamped Harmonic Oscillators

Lecture 9

Now we are ready to complete our investigations of the mass-spring differential equation $y'' + by' + ky = 0$. In the last lecture, we saw that there is an exponential solution of the form e^{st} when s is a root of the characteristic equation $s^2 + bs + k = 0$. But this quadratic equation may have complex roots, so the question is what happens to solutions when this occurs? Keep in mind that we also saw that the general solutions of the undamped equation $y'' + y = 0$ involved sines and cosines.

So how did we end up with trigonometric solutions to this differential equation instead of exponentials? Our differential equation is $y'' + y = 0$, and a guess of e^{st} yields the characteristic equation $s^2 + 1 = 0$. The roots of this equation are the imaginary number $i = \sqrt{-1}$ and its negative, so we have a solution of the form e^{it}. What is this complex expression? Recall from calculus that we can express certain functions using their power series expansions. That is, assuming our function $f(t)$ is infinitely differentiable at $t = 0$, we can write $f(t)$ as below.

$$f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2!} + f^{(3)}(0)\frac{t^3}{3!} + f^{(4)}(0)\frac{t^4}{4!} + \cdots$$

In the case of e^{it}, we can write this expression in a power series as below.

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots \\ &= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} + \cdots \end{aligned}$$

Collecting the terms that involve the constant i and those that do not yields

$$e^{it} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \right).$$

But look at the right-hand side here: The first power series is just that of the cosine function, while the second series is the sine function. That is, we have one of the most famous and beautiful of all formulas in mathematics, **Euler's formula**.

$$e^{it} = \cos(t) + i\sin(t)$$

Letting $t = \pi$ in this formula, we get every mathematician's favorite formula: $e^{i\pi} + 1 = 0$.

Now back to the differential equation $y'' + y = 0$. We have the complex solution

$$e^{it} = \cos(t) + i\sin(t).$$

But our differential equation was a real differential equation; there were no complex numbers in sight. This means that both the real part of our complex solution, $\cos(t)$, and the imaginary part of the solution, $\sin(t)$, are also solutions. This is what we saw before: Our general solution was the combination of the real and the imaginary parts of our complex solution:

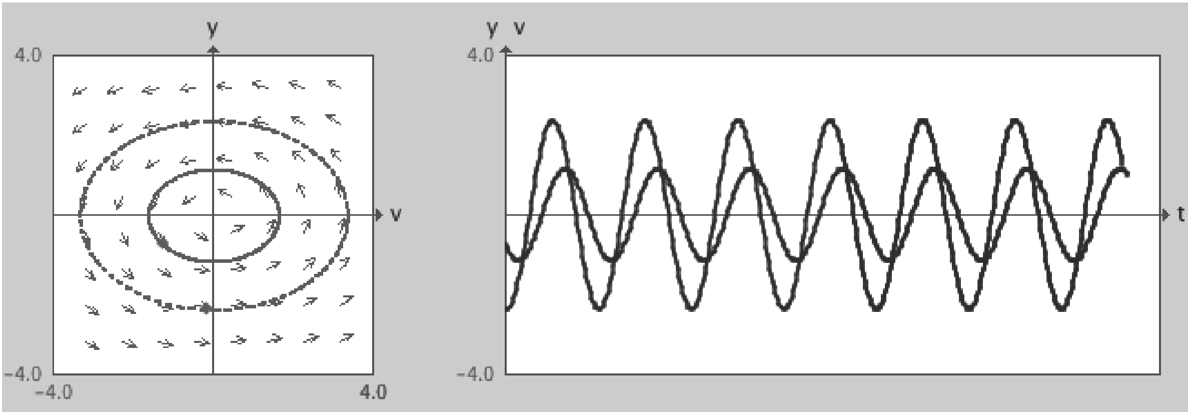
$$y(t) = k_1\cos(t) + k_2\sin(t).$$

More generally, for the differential equation $y' + ky = 0$, we get the general solution

$$y(t) = k_1 \cos(\sqrt{k}t) + k_2 \sin(\sqrt{k}t),$$

whose behavior is also periodic.

Figure 9.1

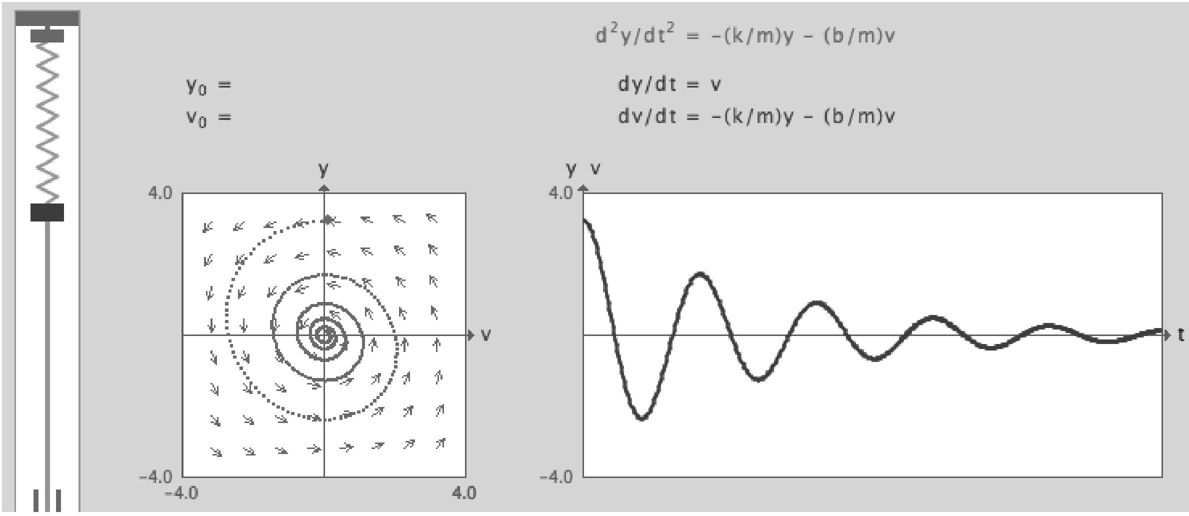


Now consider the mass-spring system with spring constant $k = 1.01$ and small damping constant $b = 0.2$. The characteristic equation is $s^2 + 0.2s + 1.01 = 0$, and the roots are the complex numbers $-0.1 + i$ and $-0.1 - i$. So we have a complex solution of the form $y(t) = e^{(-0.1+i)t}$. We can use properties of the exponential function and write it in the form $y(t) = e^{-0.1t}e^{it}$. So by Euler's formula, we now have complex solutions given by $e^{-0.1t}(\cos(t) + i\sin(t))$. Taking the real and imaginary parts of this solution gives the general solution

$$k_1e^{-0.1t}\cos(t) + k_2e^{-0.1t}\sin(t).$$

Now we have a decaying exponential as part of our solution. So the solution does oscillate around its rest position, but now the amplitude of the oscillation (the up and down height of the $y(t)$ graph) tends to zero as time goes on. We see that the mass does tend to its rest position, but it oscillates back and forth as it does so. This is the underdamped harmonic oscillator.

Figure 9.2



More generally, when we have complex roots of the characteristic equation, say $a + ib$ and $a - ib$ (here a is necessarily negative), we get the general solution

$$k_1 e^{at}\cos(bt) + k_2 e^{at}\sin(bt),$$

which similarly oscillates down to its rest position.

Recall that there was one other case of the harmonic oscillator that we have not yet solved, the case for which we found only one root for the characteristic equation. This is the **critically damped mass-spring** case. For example, consider the mass-spring system

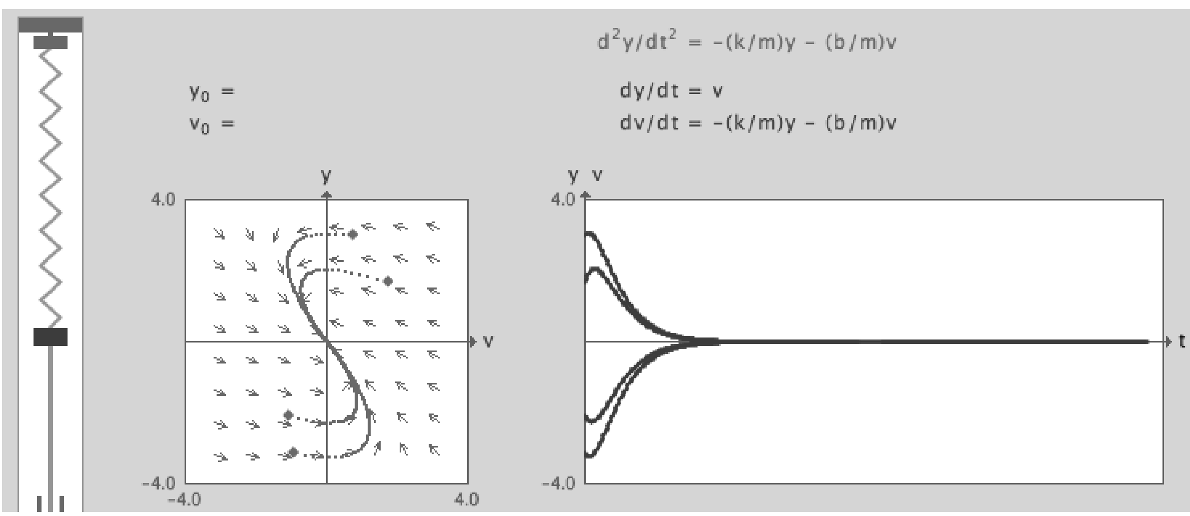
$$y'' + 2y' + y = 0.$$

The characteristic equation is now $s^2 + 2s + 1$, which factors into $(s + 1)^2 = 0$. So we have a pair of real roots given by $s = -1$. We know one solution, namely, e^{-t}. As before, we can multiply this expression by any constant to find other solutions of the form k_1e^{-t}. What about other solutions? We can easily check that $y(t) = te^{-t}$ also solves this differential equation and that, moreover,

$$k_1e^{-t} + k_2te^{-t}$$

is the general solution. What happens to the solution given by te^{-t} as time goes on? Using l'Hopital's rule, we find that the limit of the expression te^{-t} also tends to zero. Thus we see that both terms in the general solution go to zero—and in fact, in the phase plane, they do so just as in the overdamped case.

Figure 9.3



Important Terms

Euler's formula: This incredible formula provides an interesting connection between exponential and trigonometric functions, namely: The exponential of the imaginary number ($i \cdot t$) is just the sum of the trigonometric functions $\cos(t)$ and $i \sin(t)$. So $e^{(it)} = \cos(t) + i \sin(t)$.

critically damped mass-spring: A mass-spring system for which the damping force is just at the point where the mass returns to its rest position without oscillation. However, any less of a damping force will allow the mass to oscillate.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 2.1 and 3.6.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 3.4.

Roberts, *Ordinary Differential Equations*, chap. 6.1.

Strogatz, *Nonlinear Dynamics and Chaos*, chaps. 5.1 and 7.6.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Mass Spring, RLC Circuits.

Problems

1. Use the product rule to compute the derivative of $e^{2t}\cos(3t)$.

2. Use Euler's formula to expand $e^{(2t + 3it)}$.

3. Solve the quadratic equation $s^2 + bs + k = 0$, where b and k are constants.

4. The point $(\sin(t), \cos(t))$ lies on the unit circle in the plane. In which direction does it move as t increases?

5. Is the mass-spring system $y'' + 5y' + 6y$ overdamped or underdamped?

6. Use l'Hopital's rule from calculus to verify that the limit as t approaches infinity of te^{-t} is 0.

7. Find the general solution of the mass-spring system with spring constant 2 and damping constant 2.

8. If we start the mass-spring system in problem 7 off from its rest position but with velocity equal to 1, what is the maximum distance the center of mass will move?

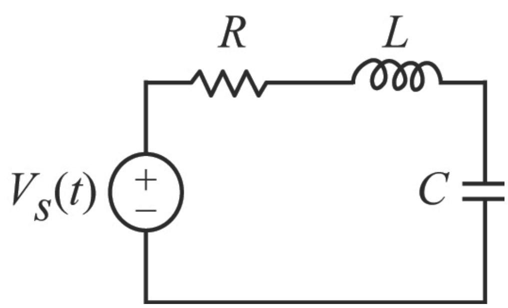
9. Consider the mass-spring systems with spring constant 1 and damping constant b. For which values of b is this system underdamped, overdamped, undamped, or critically damped?

10. We know the general solution of the undamped harmonic oscillator given by $y'' + y = 0$. What happens if we apply a constant force equal to 1 to this system? That is, what is the general solution of the equation $y'' + y = 1$? And what is the motion of the corresponding mass-spring?

Exploration

One of the simplest examples for electric circuit theory is the RLC circuit,

Figure 9.4



where R is the resistance, L is the inductance, C is the capacitance, and $V_S(t)$ denotes the voltage at the source. The differential equation for the voltage across the capacitor $v(t)$ is similar to our mass-spring example and is given by

$$LCv'' + RCv' + v = V_S(t).$$

Find the solution to this equation when $V_S(t)$ is zero. Compare the behavior of these solutions to those of the mass-spring system.

Beating Modes and Resonance of Oscillators

Lecture 10

In this lecture, we consider the periodically forced harmonic oscillator. That is, instead of hanging our spring to the ceiling, we now move the spring up and down periodically. So our differential equation becomes

$$y'' + by' + \mathrm{k}y = G(t),$$

where $G(t)$ is some periodic function. For simplicity, let us take $G(t) = \cos(wt)$. So the period of the forcing term is $2\pi/w$. There are 2 very different cases to consider. The first is when the damping constant is nonzero. In this case, if there were no periodic forcing, our mass-spring system would settle down to rest. But with the periodic forcing present, the mass cannot end up in its rest position; rather, the mass eventually begins to oscillate with period $2\pi/w$.

As an example, consider the differential equation

$$y'' + 3y' + 2y = \cos(t).$$

Solving the homogeneous equation (the equation with 0 replacing $\cos(t)$) yields the solution

$$y(t) = k_1 e^{-2t} + k_2 e^{-t}$$

since the characteristic equation is $s^2 + 3s + 2 = (s + 2)(s + 1)$. So the unforced oscillator tends to rest just as we expected.

To find one solution of the forced mass-spring system, we need to make a guess. Clearly, we cannot guess just $A\sin(t)$, since the y' term will give us a cosine term. Similarly, we cannot guess $B\cos(t)$. So the appropriate guess is $A\sin(t) + B\cos(t)$. Plugging this guess into the equation and doing a little algebra yields the particular solution with $A = 3/10$ and $B = 1/10$. So any solution of this equation is of the form

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} + \tfrac{3}{10}\sin(t) + \tfrac{1}{10}\cos(t)\,.$$

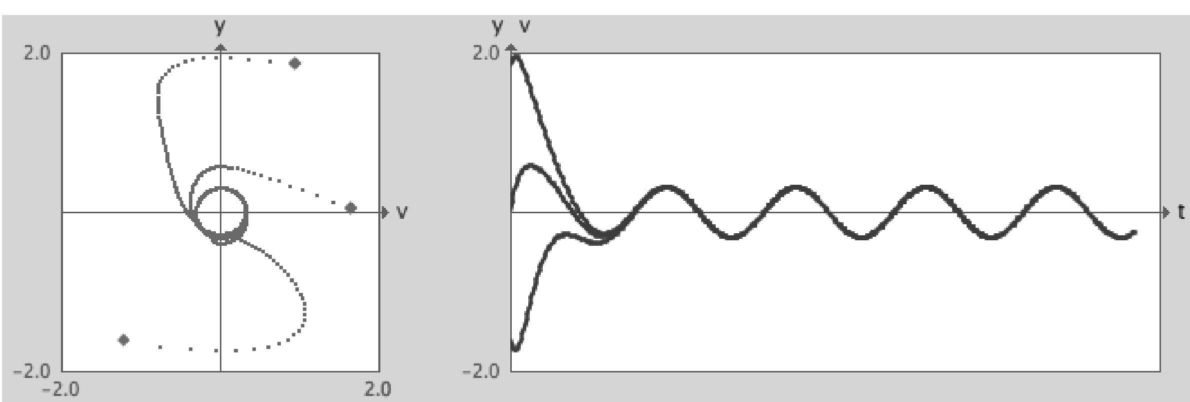
No matter which values of k_1 and k_2 we take, the corresponding solution ends up looking like

$$y(t) = \tfrac{3}{10}\sin(t) + \tfrac{1}{10}\cos(t)\,.$$

This equation has period 2π, which is exactly the period of our forcing term. This is called the **steady-state solution**.

Here are the graphs of 3 different solutions, together with the solution curves in the phase plane. Note how all solutions quickly approach the steady-state solution.

Figure 10.1



Now let's consider the case where there is no damping term and our forcing term is $\cos(wt)$, so we have a parameter present. For simplicity, consider the equation

$$y'' + y = \cos(wt).$$

Since we have no damping, the homogeneous equation is $y'' + y = 0$. So the general solution is $k_1\cos(t) + k_2\sin(t)$. This solution does not tend to rest. To get the full solution, we make the guess $A\sin(wt) + B\cos(wt)$. Plugging this into the differential equation, we find

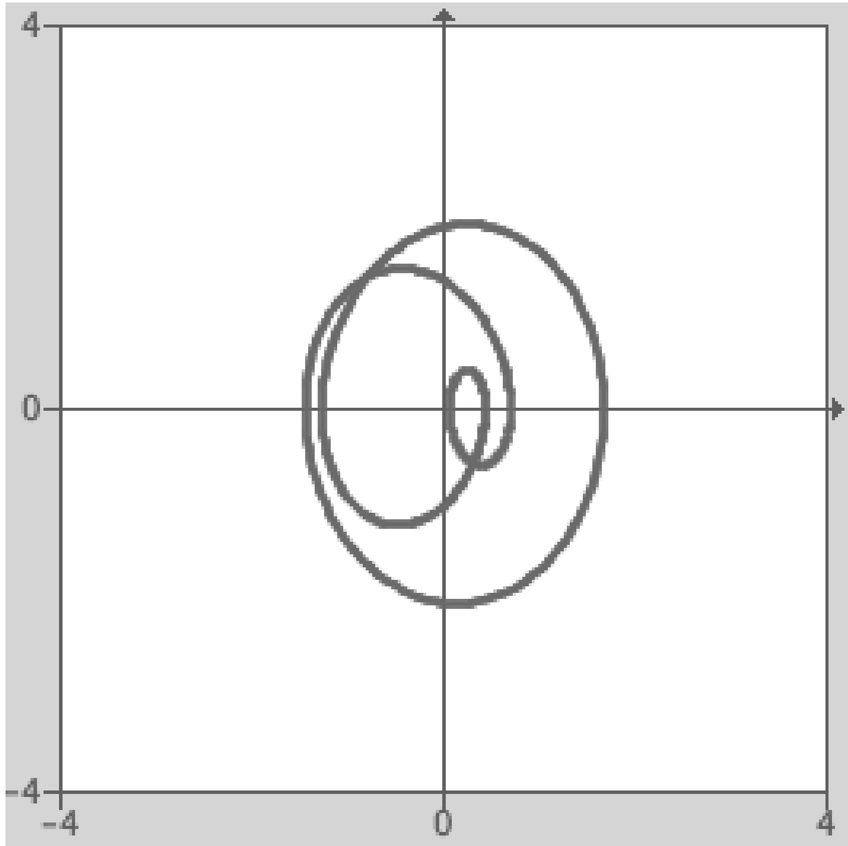
$$B(1 - w^2)\cos(wt) + A(1 - w^2)\sin(wt) = y'' + y = \cos(wt).$$

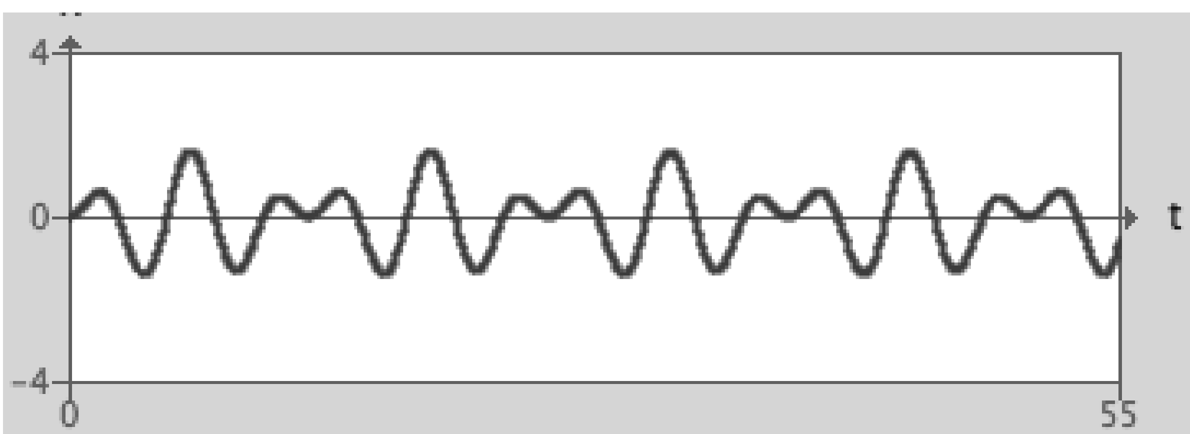
So we must have $B = 1/(1 - w^2)$ and $A = 0$. That is, our particular solution is

$$\frac{1}{1-w^2}\cos(wt) + k_1\cos(t) + k_2\sin(t).$$

But wait a moment—this is fine as long as w is not equal to 1; if $w = 1$, this is no longer the solution. That is, if the period of the forcing is the same as the natural period of the spring, we are in trouble. We'll deal with this case later. But first look what happens for different values of w. Here are some graphs of the solution that satisfies $y(0) = y'(0) = 0$. First, when $w = 1.5$, we see a periodic motion, both in the y-v plane and as the graph of $y(t)$.

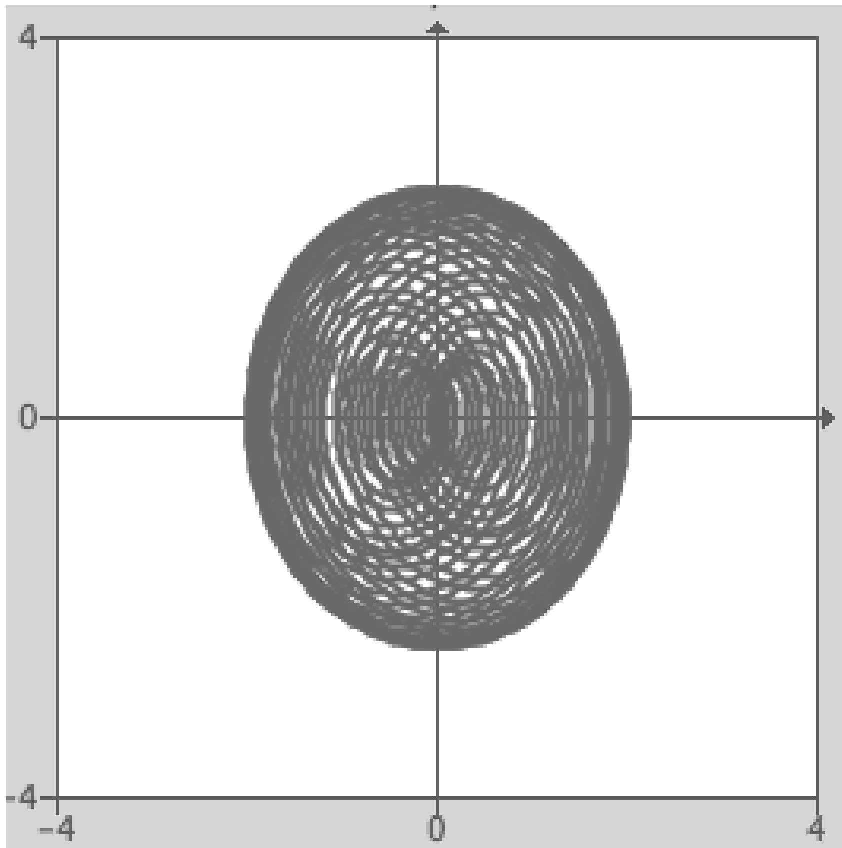
Figure 10.2

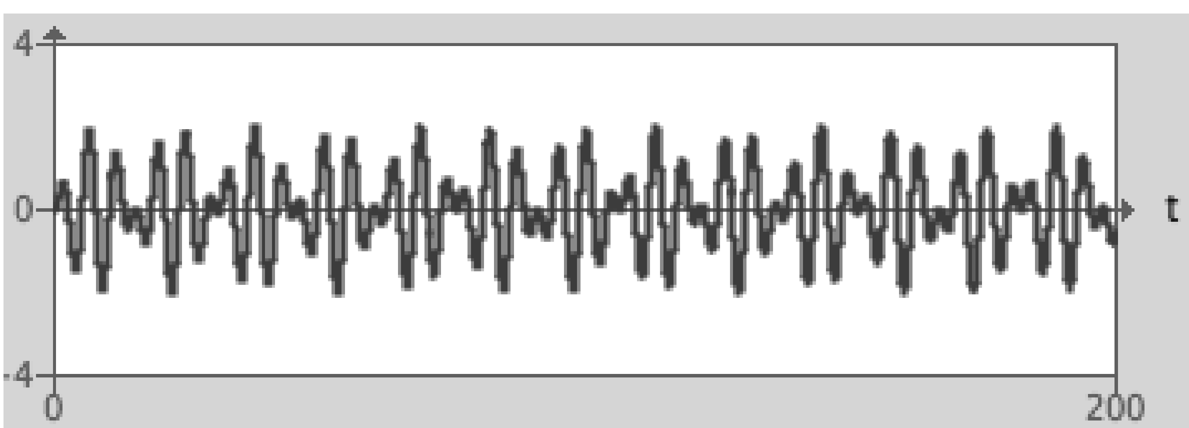




But when $w = \sqrt{2}$, the solution is no longer periodic.

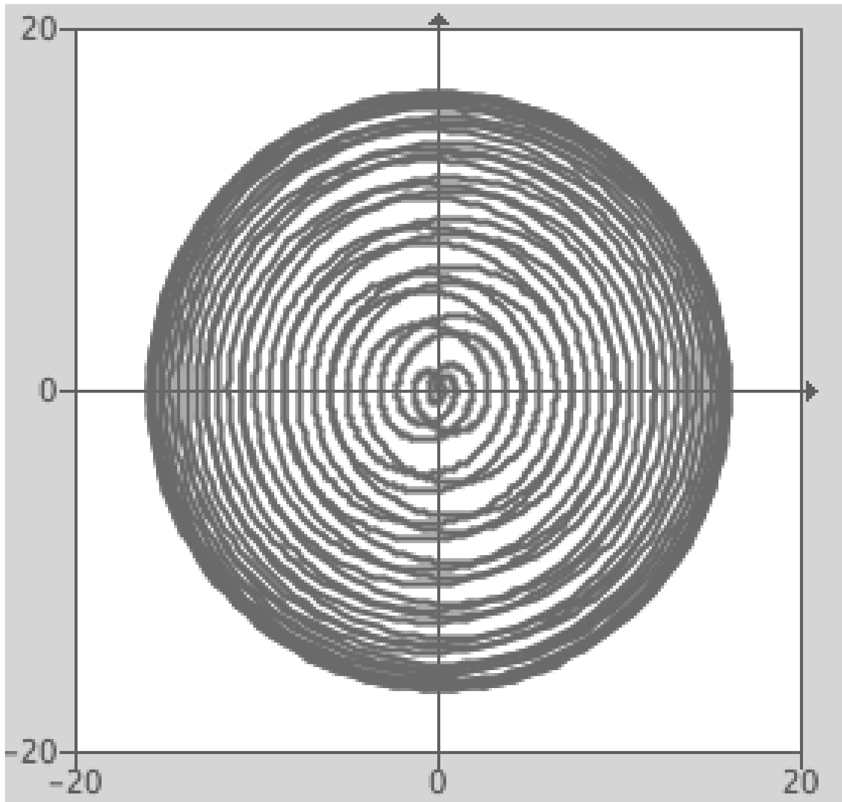
Figure 10.3

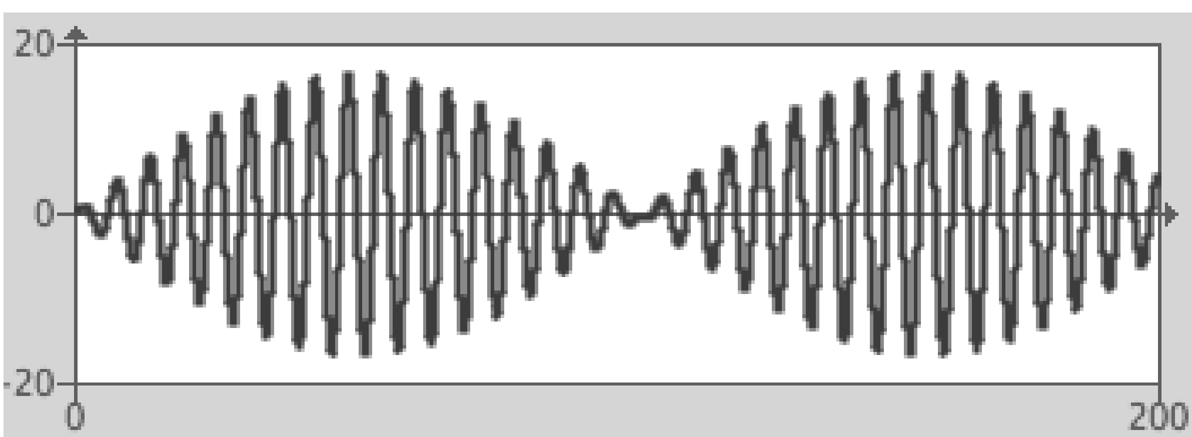




And when w gets very close to 1, say $w = 1.06$, we get **beating modes**.

Figure 10.4





Note that the y-coordinate runs here from −20 to 20. So the mass sometimes undergoes large oscillations and sometimes does not. Why? Since w is very close to 1, the natural period and the forcing period are close together. This means that for a long period we are forcing the spring exactly the way it naturally wants to go, so the oscillations become larger and larger. But eventually, $\cos(t)$ and $\cos(wt)$ are relatively far apart, and this means that we are forcing in the direction opposite to that the spring wants to go, resulting in smaller and smaller oscillations.

Now let's go back to the case $w = 1$, that is, the differential equation

$$y'' + y = \cos(t).$$

We no longer have our old solution

$$\frac{1}{1-w^2}\cos(wt) + k_1\cos(t) + k_2\sin(t)$$

since w is now equal to 1. Indeed, we cannot have a solution of the form $A\cos(t) + B\sin(t)$ since that expression solves the homogeneous equation. So what to do? How about making the guess $y(t) = At\cos(t) + Bt\sin(t)$? Plugging this into the differential equation then yields

$$y'' + y = -2A\sin(t) + 2B\cos(t).$$

Setting this equal to $\cos(t)$ tells us that $B = 1/2$ while $A = 0$. So our general solution now becomes

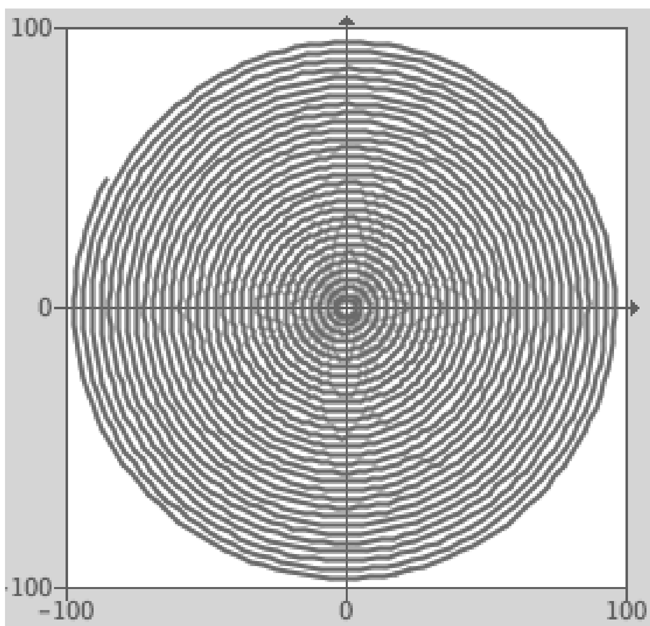
$$\tfrac{1}{2}t\sin(t) + k_1\cos(t) + k_2\sin(t)\ .$$

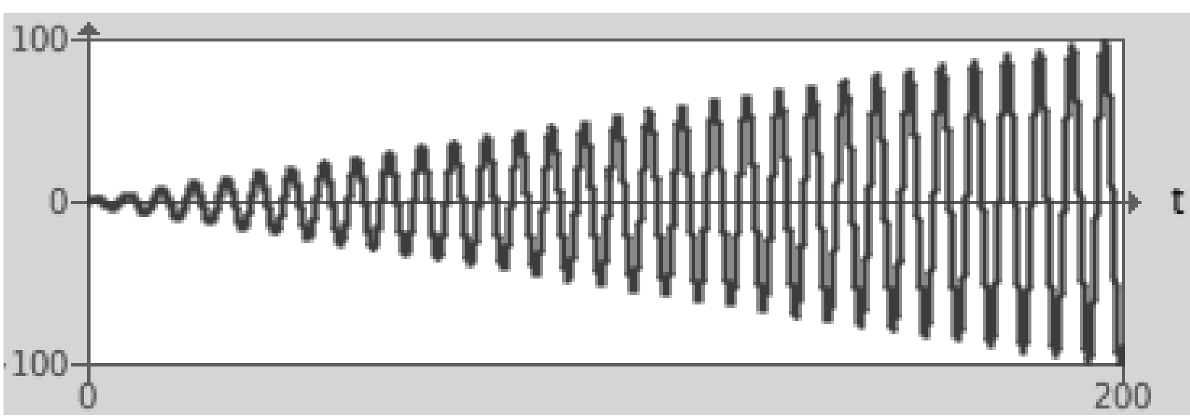
Therefore the solution that satisfies $y(0) = y'(0) = 0$ is just

$$y(t) = \tfrac{1}{2} t \sin(t) .$$

Note that this expression then oscillates back and forth with increasing amplitude; that is, our spring keeps moving alternately higher and lower. Note that in the graph below, the y-values extend from −100 to 100.

Figure 10.5





This is the phenomenon called **resonance**. We keep pushing and pulling our mass-spring system in just the way it wants to go, so the oscillations keep getting larger and larger, eventually leading to disaster.

Some experts blame resonance for the collapse of the Tacoma Bridge (a.k.a. "Galloping Gertie") back in 1940, just 4 months after the bridge was opened to traffic. A similar phenomenon led to the closing of London's Millennium Bridge in June 2000, just 2 days after it was opened to pedestrian traffic. Indeed, when crossing a bridge, soldiers no longer march in step for fear that their steps will coincide with the resonant frequency of the bridge.

Important Terms

beating modes: The type of solutions of periodically forced and undamped mass-spring systems that periodically have small oscillations followed by large oscillations.

resonance: The kind of solutions of periodically forced and undamped mass-spring systems that have larger and larger amplitudes as time goes on. This occurs when the natural frequency of the system is the same as the forcing frequency.

steady-state solution: A periodic solution to which all solutions of a periodically forced and damped mass-spring system tend.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 4.3 and 4.5.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 6.2.

Roberts, *Ordinary Differential Equations*, chap. 6.1.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 7.6.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Beats and Resonance, Mass Spring.

Problems

1. **a.** Solve the periodically forced first-order equation

$$y' + y = \sin(t).$$

   b. What happens to all solutions of this equation?

   c. Sketch the graphs of some of these solutions in the region $0 < t < 20$.

   d. What happens to solutions if we change this equation to

$$y' - y = \sin(t)?$$

   e. What happens to these solutions as time moves backward?

2. Find the steady-state solution of the forced mass-spring system with spring constant 2, damping constant 1, and forcing term $\sin(t)$.

3. What is the general solution of the forced mass-spring system given by $y'' + y = e^{-t}$?

4. Consider the mass-spring system $y'' + 3y = \sin(wt)$. For which value of w is this system in resonance?

5. For which values of w is the function $\cos(wt) + \cos(t)$ a periodic function?

6. Consider the undamped and forced mass-spring system $y'' + y = \cos(wt)$. Using the results of the previous problem, determine the values of w for which the solutions of this system are periodic functions.

Exploration

Consider the case of a pair of undamped harmonic oscillators.

$$y_1'' = -w_1^2 y_1$$

$$y_2'' = -w_2^2 y_2$$

These equations are decoupled, so we can easily solve both. But let's view these solutions a little differently. The quantity w_2/w_1 is called the frequency ratio. First show that the pair of solutions of these 2 equations is periodic if and only if the frequency ratio is a rational number. Plot a solution $(y_1(t), y_2(t))$ in the plane when, say, $w_1 = 2$ and $w_2 = 5$. What if $w_1 = 1$ and $w_2 = 2^{1/2}$? Change these equations to polar coordinates (r_j, θ_j) for $j = 1, 2$. What is the new system of differential equations? Do you see that $r_j' = 0$? So we can now think of solutions as residing on a torus (i.e., the surface of a doughnut). What would these solutions look like on the torus?

Linear Systems of Differential Equations

Lecture 11

We now move on to linear systems of differential equations. Much of the theory surrounding linear systems of differential equations involves tools from the area of mathematics known as linear algebra, including matrix arithmetic, determinants, eigenvalues, and eigenvectors.

Let's consider 2-dimensional, autonomous linear systems. These are systems of differential equations of the form

$$x' = ax + by$$

$$y' = cx + dy,$$

where a, b, c, and d are all parameters. We will often write this in matrix form $Y' = AY$, where Y is the 2-dimensional vector

$$Y = \begin{pmatrix} x \\ y \end{pmatrix}$$

and A is the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The product of the matrix A and the vector Y is the new vector whose first entry is the dot product of the first row of A with Y and whose second entry is the dot product of the second row of A with the vector Y—that is, the vector

$$AY = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

A solution of such a linear system is then a vector depending on t (a curve in the plane) given by

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $x(t)$ and $y(t)$ are the solutions of the given system. So $x'(t) = ax(t) + by(t)$, and $y'(t) = cx(t) + dy(t)$.

For example, consider the linear system

$$Y' = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} Y = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can easily check that one solution is $x(t) = e^{2t}$, $y(t) = e^{2t}$. We write this solution in vector form as

$$Y(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

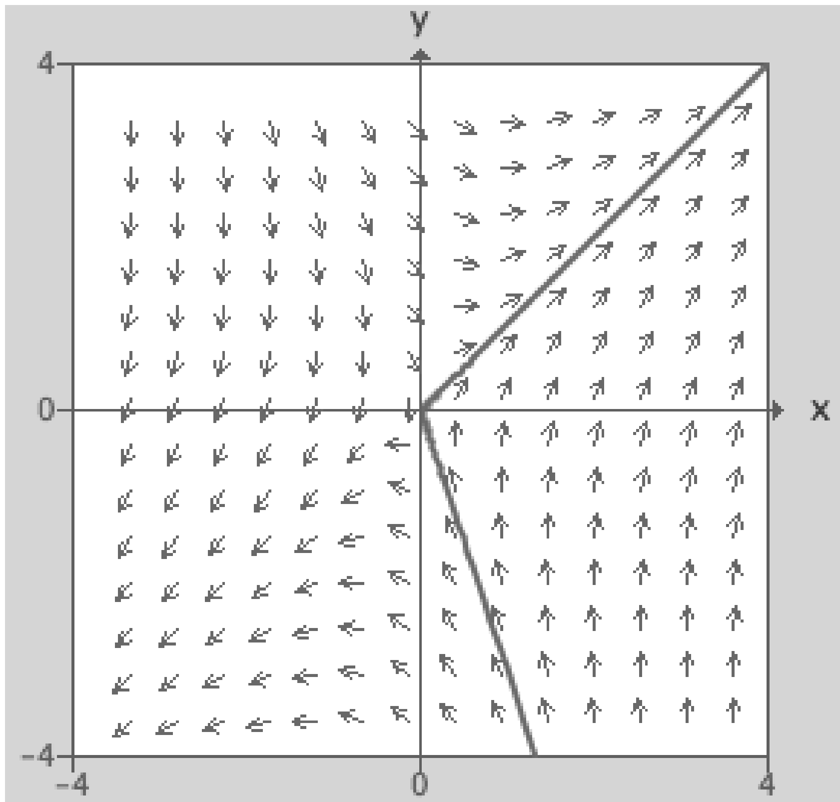
Note that this is a straight line solution of the system; in the phase plane, this solution lies along the straight line $y = x$. It tends to the origin as time goes to $-\infty$ and to ∞ as time goes to ∞.

Another solution is $x(t) = e^{-2t}$, $y(t) = 3e^{-2t}$, or in vector form,

$$Y(t) = \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Again, this is a straight line solution, this time lying on the line $y = -3x$ in the phase plane and tending to the origin as time increases. These solutions in the phase plane look as follows.

Figure 11.1

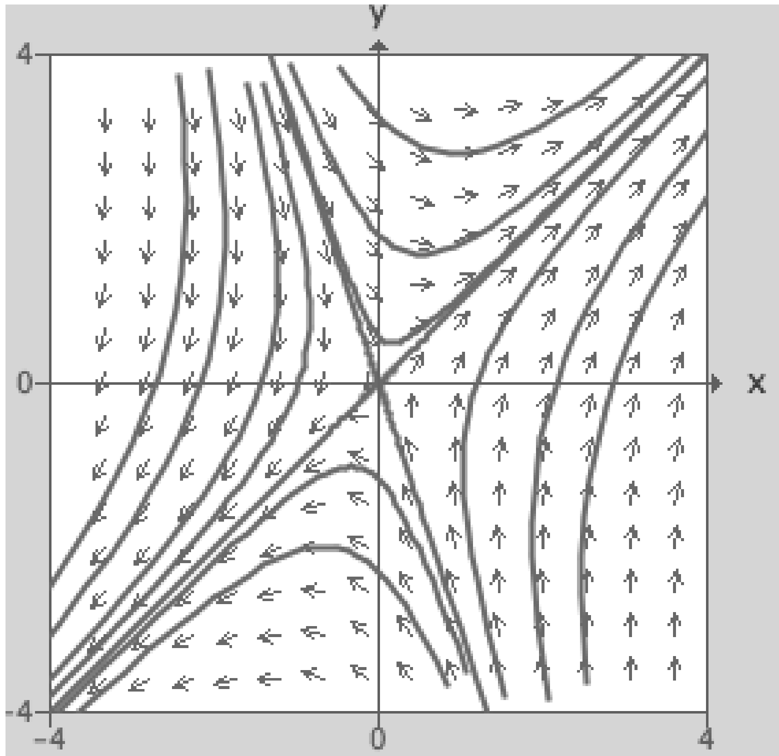


As earlier, any expression of the form

$$Y(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

for any values of c_1 and c_2 is also a solution. Here is a collection of such solutions in the phase plane.

Figure 11.2



We say that the equilibrium point at the origin in this phase plane is a **saddle point**.

In fact,

$$Y(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

is the general solution to this differential equation. In order to see why this is true, we must be able to determine values of c_1 and c_2 that solve any initial value problem of the form

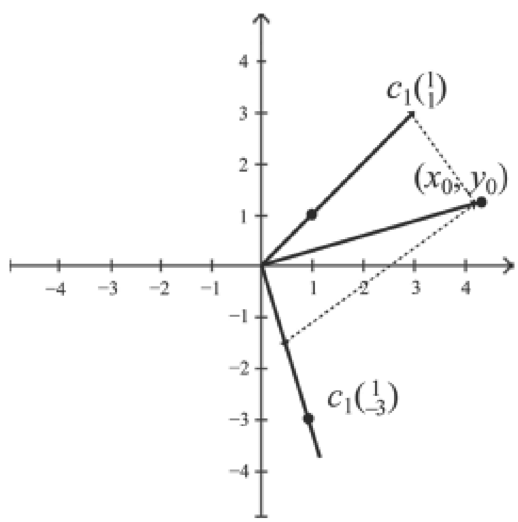
$$Y(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

That is, we must be able to find c_1 and c_2 so that

$$Y(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Notice that the 2 vectors (1, 1) and (1, −3) are linearly independent. That is, they do not lie along the same straight line passing through the origin. This means that by the parallelogram rule, we can always stretch or contract these vectors (by multiplying by c_1 and c_2) so that the resulting vectors add up to (x_0, y_0).

Figure 11.3



$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Whenever we have 2 such solutions whose initial vectors are linearly independent, a similar combination of them gives us the general solution.

As another example, consider

$$Y' = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} Y .$$

We can easily confirm that the only equilibrium solution is at the origin. To find the other solutions, note that the first equation reads $x' = -x$. So we know that the general solution here is k_1e^{-t}. Then the second equation is $y' = -2y + k_1e^{-t}$, which is a first-order linear and nonhomogeneous equation. We know how to solve this. The solution is $y(t) = k_2e^{-2t} + k_1e^{-t}$. So our solutions to the linear system are

$$x(t) = k_1e^{-t}$$

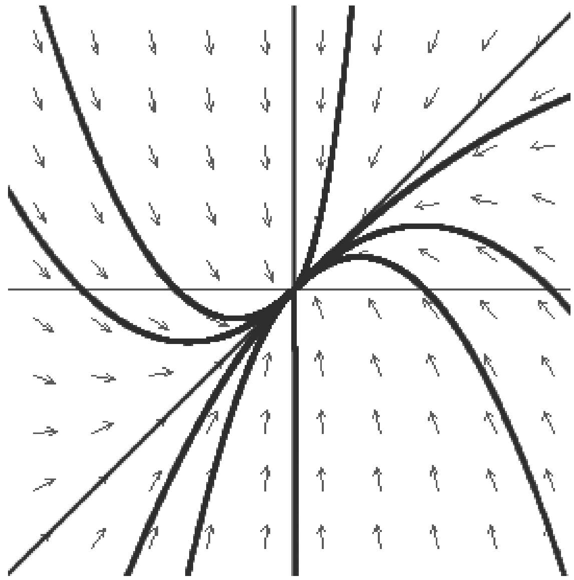
$$y(t) = k_2e^{-2t} + k_1e^{-t}.$$

In vector form, this solution is as below.

$$Y(t) = k_1e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When $k_1 = 0$ and $k_2 = 1$, we get the straight line solution $x(t) = 0$ and $y(t) = e^{-2t}$ lying along the x-axis. And when $k_1 = 1$ and $k_2 = 0$, we get another straight line solution $x(t) = e^{-t}$, $y(t) = \mathrm{e}^{-t}$, this time lying along the line $y = x$. It is the same as before, but now both straight line solutions tend inward to the origin. We see that the phase plane is a sink.

Figure 11.4



Thus we see again 2 straight line solutions for this system. This will be the way we proceed to solve linear systems; we'll first hunt down some straight line solutions and then use them to put together the general solution.

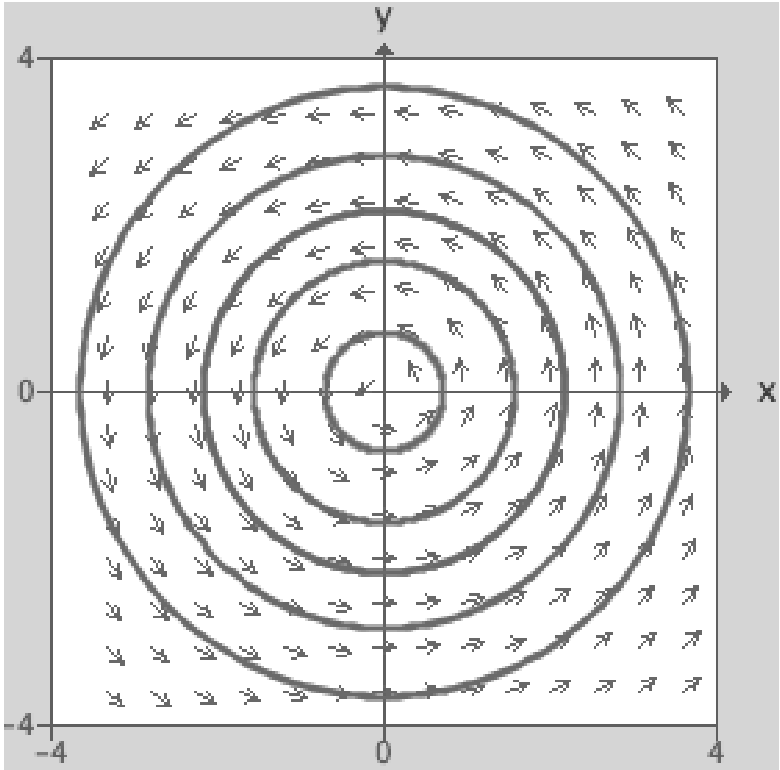
This straight line method does not always produce straight line solutions, however. There are many other possible phase planes for linear systems, which we produce with the same method.

For example, for the system

$$Y' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y,$$

we have $x'(t) = -y(t)$ while $y'(t) = x(t)$. So some solutions are $x(t) = c\cos(t)$, $y(t) = c\sin(t)$ for any constant c. These solutions all lie along circles in the phase plane, and the equilibrium point at the origin is now called a **center**.

Figure 11.5



Let's look at a different example.

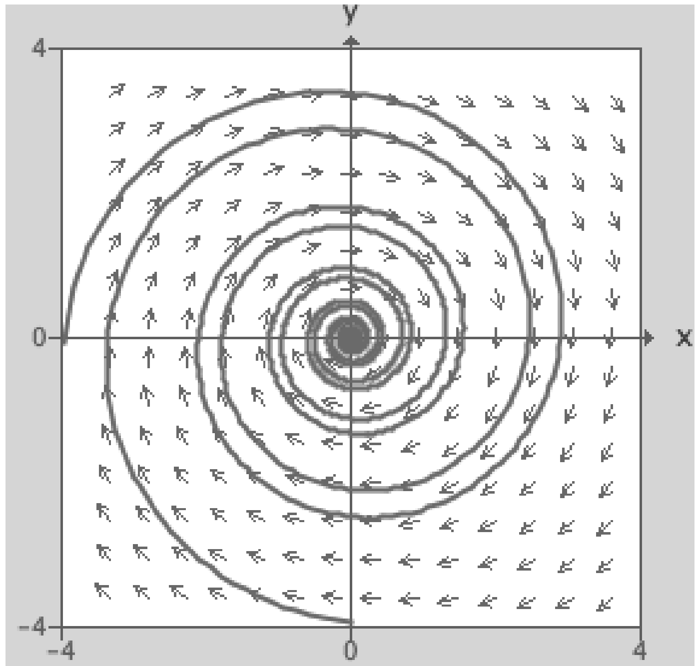
$$Y' = \begin{pmatrix} -0.1 & 1 \\ -1 & -0.1 \end{pmatrix} Y$$

We easily determine that some solutions are of the form

$$Y(t) = ce^{-0.1t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}.$$

Without the exponential term, these solutions would all lie along circles as above. But, since we have the exponential of $-0.1t$, these solutions must now spiral in toward the origin as time increases. Our phase plane now looks as follows, and this equilibrium solution is called a **spiral sink**.

Figure 11.6



The first question concerning linear systems is what are the equilibrium points. Certainly the origin, (0, 0), is always an equilibrium point, but are there others? If so, they would be solutions to the system of linear algebraic equations

$$ax + by = 0$$

$$cx + dy = 0.$$

The first equation says that $x = -(b/a)y$ (as long as a is nonzero). Substituting this into the second equation, we find

$$(ad - bc)y = 0.$$

So there are only 2 possibilities here, either $y = 0$ or $ad - bc = 0$. In the first case, we then have that x is also 0, so we get the equilibrium point at the origin we already know about. In the other case, $ad - bc = 0$, we then have that any y value works; so $x = -(b/a)y$. So this yields a straight line of equilibrium solutions given by $x = -(b/a)y$.

A similar argument works if $a = 0$, at least as long as one of the coefficients in the matrix is nonzero. If all the entries in the matrix equal 0, we have a pretty simple system—namely, $x' = 0$ and $y' = 0$, so all points (x, y) are equilibrium points. We'll neglect this simple case in the future.

So, to summarize, for the linear system of ODEs $Y' = AY$, we have

1. A straight line of equilibrium points if $ad - bc = 0$
2. A single equilibrium point at the origin if $ad - bc \neq 0$.

The quantity $ad - bc$ will reappear often in the sequel; it is called the **determinant** of the matrix. We abbreviate it $\det A = ad - bc$.

Notice also that a linear system of algebraic equations

$$ax + by = 0$$

$$cx + dy = 0$$

has nonzero solutions if and only if $\det A = 0$.

Important Terms

center: An equilibrium point for a system of differential equations for which all nearby solutions are periodic.

determinant: The determinant of a 2-by-2 matrix is given by $ad - bc$, where a and d are the terms on the diagonal of the matrix, while b and c are the off-diagonal terms. The determinant of a matrix A ($\det A$, for short) tells us when the product of matrix A with vector Y to give zero ($A\ Y = 0$) has only one solution (namely the 0 vector, which occurs when the determinant is non-zero) or infinitely many solutions (which occurs when the determinant equals 0).

saddle point: An equilibrium point that features one curve of solutions moving away from it and one other curve of solutions tending toward it.

spiral sink: An equilibrium solution of a system of differential equations for which all nearby solutions spiral in toward it.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 3.1.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 2.4.

Kolman and Hill, *Elementary Linear Algebra*, chap. 3.

Lay, *Linear Algebra*, chap. 3.

Roberts, *Ordinary Differential Equations*, chap. 7.

Strang, *Linear Algebra and Its Applications*, chap. 4.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 5.1.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Linear Phase Portraits.

Problems

1. What is the product of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$?

2. **a.** Write the system of equations

$$x' = y$$

$$y' = -x$$

in matrix form.

b. What are the equilibrium points for the above system?

3. Are the vectors $(1, 2)$ and $(-4, -8)$ linearly independent?

4. **a.** Is

$$\begin{pmatrix} e^{2t} + e^{t} \\ e^{2t} \end{pmatrix}$$

a solution of the system

$$x' = x + y$$

$$y' = 2y?$$

b. Is $\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$ a solution of the above equation?

5. Consider the linear system below.

$$x' = x$$

$$y' = x + 2y$$

a. Write this system in vector form.

b. Since the system can be decoupled, first find the solution of $x' = x$ and then use that to find the solution of $y' = x + 2y$.

c. Write this solution in vector form.

d. Do the 2 above solutions generate the general solution?

e. Sketch the above solutions in the phase plane.

Exploration

We have seen that the determinant of a 2 × 2 matrix allows us to decide when a system of 2 linear equations has a nonzero solution. What about the case of 3 linear equations, such as the below?

$$a_{11}x + a_{12}y + a_{13}z = 0$$

$$a_{21}x + a_{22}y + a_{23}z = 0$$

$$a_{31}x + a_{32}y + a_{33}z = 0$$

Can you find a formula (depending on the a_{ij}) that tells you when this system of equations has nonzero solutions?

An Excursion into Linear Algebra

Lecture 12

Recall the linear system that we solved last time:

$$Y' = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} Y \, .$$

There were 2 straight line solutions given by

$$Y_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } Y_2(t) = e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

$Y_1(t)$ lies along the diagonal line $y = x$, and $Y_2(t)$ lies along the line $y = -3x$. Note that

$$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

We'll see these equations again.

How do we solve the general linear system? As above, we have often seen that there are solutions of a system that run along a straight line. How do we find these solutions? Well, we would need the vector field to either point straight in toward the origin or point directly away from the origin. That is, we would need the right-hand side of our system, namely AY, to be some multiple of Y, say λY. That is, we would need to find a vector Y such that $AY = \lambda Y$ or $AY - \lambda Y = 0$, where 0 is the zero vector—that is, (0, 0). Note that this is precisely what is happening in our example above.

We usually write this latter equation as $(A-\lambda I)Y=0$, where I is the identity matrix

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Written out, these equations become

$$\begin{aligned}(a-\lambda)x+by&=0\\ cx+(d-\lambda)y&=0.\end{aligned}$$

So to find places where the vector field points toward or away from the origin, we must find a value of λ for which the above equations have a nonzero solution. Such a λ-value is called an **eigenvalue** for our system, and any nonzero solution of the equation $(A-\lambda I)Y=0$ is called an **eigenvector** corresponding to λ.

So how do we determine the eigenvalues? We need to find a value of λ for which the above system of algebraic equations has nonzero solutions. But we saw earlier that that happens when the determinant of the corresponding matrix is zero. So λ_0 is an eigenvalue if it is a root of the equation $\det(A-\lambda I)=0.$ Let's write that out explicitly. We find that

$$\det(A-\lambda I)=(a-\lambda)(d-\lambda)-bc=\lambda^2-(a+d)\lambda+ad-bc=0.$$

This equation is also called the characteristic equation. Note that it is a quadratic equation, so there are usually 2 distinct roots for this equation. These roots are the eigenvalues for the matrix. Then we can find the

corresponding eigenvector by simply finding a nonzero vector solution to the equation

$$(A - \lambda_0 I)\ Y = 0,$$

where λ_0 is the given eigenvalue.

By the way, note that in the characteristic equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

we see the term $ad - bc$. That's what we called the determinant of the matrix A, or det$A$. There is also the term $a + d$ (i.e., the sum of the main diagonal terms in the matrix). This expression is called the **trace** of matrix A, which we write as TrA. So the characteristic equation can be written

$$\lambda^2 - (Tr\ A)\lambda + \det A = 0\ .$$

The important observation is that if λ is an eigenvalue for the matrix A and Y_0 is its corresponding eigenvector, then we have a solution to the system given by

$$Y(t) = e^{\lambda t} Y_0,$$

which is easily checked.

For example, consider the linear system

$$Y' = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} Y\ .$$

We first find the eigenvalues: They are the roots of the characteristic equation

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} = 0 ,$$

which simplifies to $(2-\lambda)(-1-\lambda)=0$. Clearly these roots are 2 and −1. So these are our eigenvalues. Next we find the eigenvector corresponding to the eigenvalue 2. To do this, we must solve the system of algebraic equations

$$\begin{aligned} (2-2)x + y &= 0 \\ 0x + (-1-2)y &= 0 \end{aligned}$$

that reduces to $y = 0$ and $-3y = 0$. Notice that these 2 equations are redundant; this is always the case since we know that we must find a nonzero solution to the eigenvector equation. Our solution is any (nonzero) vector with y-coordinate equal to 0. So for example, one eigenvector corresponding to the eigenvalue 2 is the vector

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(we could have chosen any nonzero x-component).

So we have one solution to this system, namely

$$Y_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or $x(t) = e^{2t}$, $y(t) = 0$. Note that this is a straight line solution lying on the x-axis. It is easy to check that any constant times this solution is also a solution to our system, so we find other straight line solutions lying on the x-axis—some on the positive side and some on the negative side. But all tend to (plus or minus) infinity as time goes on and to 0 as time goes backward.

For the eigenvalue −1, we similarly find a corresponding eigenvector

$$Y = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

and thereby get the solution

$$Y_1(t) = e^{-t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

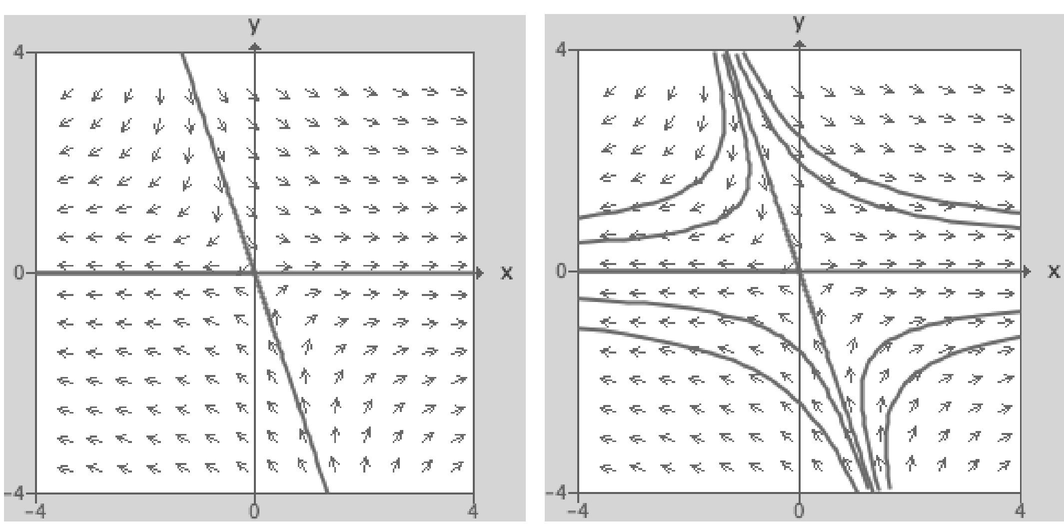
This is also a straight line solution lying along the line $y = -3x$, which tends to the origin as time goes on and away from the origin (toward infinity) as time goes backward.

So we get a whole family of solutions of the form

$$Y(t) = k_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Indeed, this turns out to be the general solution. Below are the straight line solutions together with some other solutions in the phase plane. This is an example of a phase plane that has a saddle equilibrium point at the origin.

Figure 12.1



There are 2 other possible things that can happen when we have 2 real, distinct, and nonzero eigenvalues. Either both can be positive or both can be negative. In the former case, our equilibrium point is a (real) source, since our general solution will be of the form

$$k_1 e^{\lambda_1 t} Y_1 + k_2 e^{\lambda_2 t} Y_2 ,$$

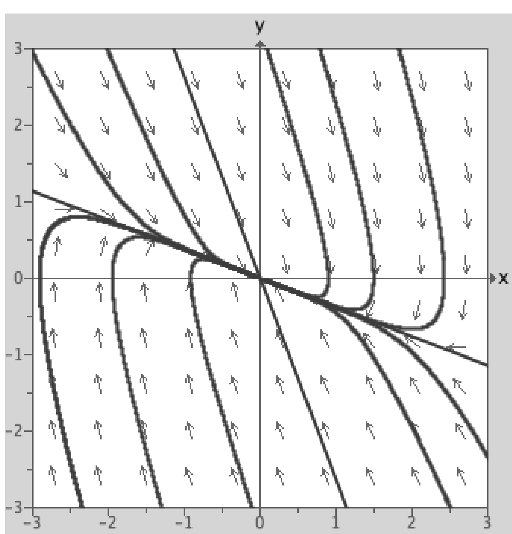
where both λ_1 and λ_2 are positive. Thus all nonzero solutions will move away from the origin. In the other case, we have a (real) sink since all solutions will now tend to the origin.

For example, if our matrix is

$$\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix},$$

then the eigenvalues are the roots of $\lambda^2 + 3\lambda + 1 = 0$, which by the quadratic formula are $\dfrac{-3 \pm \sqrt{5}}{2}$. Both of these are negative, so we get a real sink at the origin.

Figure 12.2

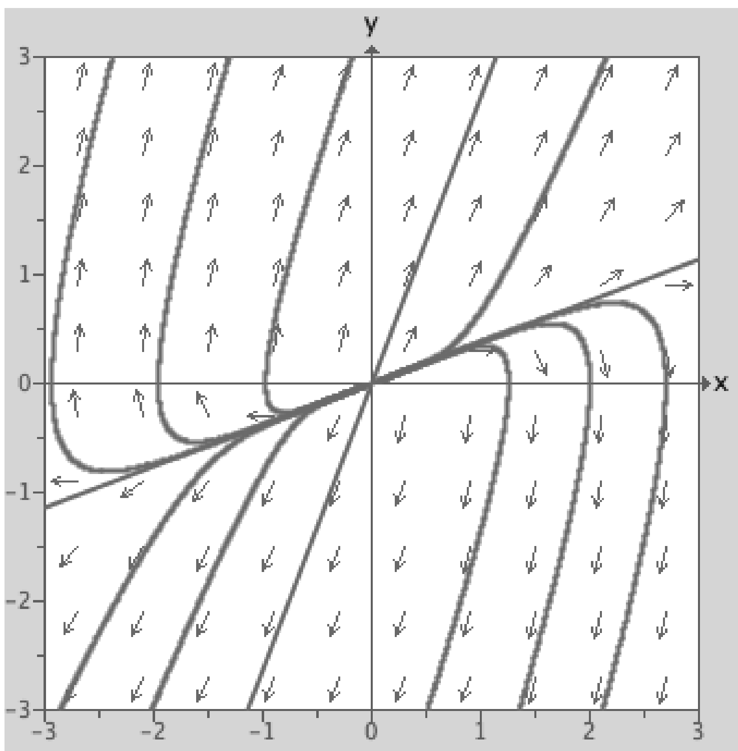


But if our matrix is

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix},$$

the eigenvalues are $\frac{3 \pm \sqrt{5}}{2}$, which are now both positive, so we get a real source.

Figure 12.3



Then there are a number of other possibilities for the eigenvalues. These roots of the characteristic equation could be complex; real, nonzero, but not distinct (i.e., repeated roots); or zero. As we shall see in upcoming lectures, each of these cases gives very different phase planes.

Important Terms

eigenvalue: A real or complex number usually represented by λ (lambda) for which a vector Y times matrix A yields non-zero vector λY. In general, an $n \times n$ matrix will have n eigenvalues. Such values (which have also been called proper values and characteristic values) are roots of the corresponding characteristic equation. In the special case of a triangular matrix, the eigenvalues can be read directly from the diagonal, but for other matrices, eigenvalues are computed by subtracting λ from values on the diagonal, setting the determinant of that resulting matrix equal to zero, and solving that equation.

eigenvector: Given a matrix A, an eigenvector is a non-zero vector that, when multiplied by A, yields a single number λ (lambda) times that vector: $AY = \lambda Y$. The number λ is the corresponding eigenvalue. So, when λ is real, AY scales the vector Y by a factor of lambda so that AY stretches or contracts vector Y (or $-Y$ if lambda is negative) without departing from the line that contains vector Y. But λ may also be complex, and in that case, the eigenvectors may also be complex vectors.

trace: The sum of the diagonal terms of a matrix from upper left to lower right.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 3.2–3.3.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 3.1.

Kolman and Hill, *Elementary Linear Algebra*, chap. 7.

Lay, *Linear Algebra*, chap. 5.

Roberts, *Ordinary Differential Equations*, chap. 7.

Strang, *Linear Algebra and Its Applications*, chap. 5.

Strogatz, *Nonlinear Dynamics and Chaos*, chaps. 5.1–5.2.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Linear Phase Portraits.

Problems

1. Compute the determinant of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$.

2. What do you know about the equilibrium points for the system

$$Y' = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} Y\ ?$$

3. Compute the trace and the determinant of the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

4. **a.** Sketch the direction field for the linear system

$$Y' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} Y .$$

b. Do you see any straight line solutions for this system?

5. What are the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ?$$

6. What are the eigenvalues of an upper triangular matrix

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix},$$

where * can be any number?

7. What are the eigenvalues and eigenvectors for the very special matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ?$$

8. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 3 \\ \sqrt{2} & 3\sqrt{2} \end{pmatrix} .$$

9. Find the general solution of

$$Y' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} Y$$

and sketch the phase plane.

10. Find the solution to the previous system satisfying the initial condition $Y(0) = (1, 0)$.

Explorations

1. Consider the linear system

$$Y' = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} Y .$$

First find both straight line solutions. Then determine how all other solutions tend to the origin as time goes on.

2. In the last exploration of linear algebra, we saw that that the determinant of a 3×3 matrix allows us to decide when a system of 3 linear equations has a nonzero solution. This should allow us to expand the notion of eigenvalue and eigenvector to this case. So find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

What would be the behavior of the solutions of the corresponding linear system of differential equations?

Visualizing Complex and Zero Eigenvalues

Lecture 13

Recall that to solve the linear system of differential equations given by

$$Y' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y ,$$

we followed a 4-step process. First we wrote down the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - T\lambda + D = 0 ,$$

where T is the trace of the matrix and D is the determinant. Second, we solved this quadratic equation to find the eigenvalues

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2} .$$

Third, for each eigenvalue $\lambda_\pm$, we found the corresponding eigenvectors $V_\pm$ by solving the system of algebraic equations below.

$$\begin{aligned} (a-\lambda)x + by &= 0 \\ cx + (d-\lambda)y &= 0 \end{aligned}$$

Finally, this gave us the general solution below.

$$k_1 e^{\lambda_- t} V_- + k_2 e^{\lambda_+ t} V_+$$

But the roots of the characteristic equation may be complex—so what happens in that case?

Let's begin with the simple example

$$Y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y .$$

We actually know how to solve this system since the system is just $x' = y$ and $y' = -x$. So we really have $x'' = -x$, and we know the solution of this second-order equation. The general solution is $x(t) = k_1\cos(t) + k_2\sin(t)$. Therefore $y(t) = x'(t) = -k_1\sin(t) + k_2\cos(t)$.

Altogether, this yields

$$Y(t) = k_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + k_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

Let's see how this solution arises out of the eigenvalue/eigenvector process. Our characteristic equation here is $\lambda^2 + 1 = 0$, so the eigenvalues are the imaginary numbers i and $-i$. To find the eigenvector corresponding to i, we must solve the system of equations

$$-ix + y = 0$$

$$-x - iy = 0.$$

These 2 equations are redundant, since multiplying the second equation by i gives the first equation. So the eigenvector corresponding to the eigenvalue i is any nonzero vector that satisfies $y = ix$. So for example, one eigenvector is

$$\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

We therefore get the solution

$$Y(t) = e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

But there is a problem here. We started with a system of real differential equations but ended up with a complex solution. Let's remedy this using Euler's formula. Our solution can be expanded via Euler to

$$Y(t) = (\cos(t) + i\sin(t))\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + i\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

This is still a complex solution to a real differential equation. But were we to plug this expression into the equation $Y' = AY$, the real part would remain real and the imaginary part would remain imaginary. That means that the real part of this complex solution,

$$Y_{real}(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix},$$

and the imaginary part,

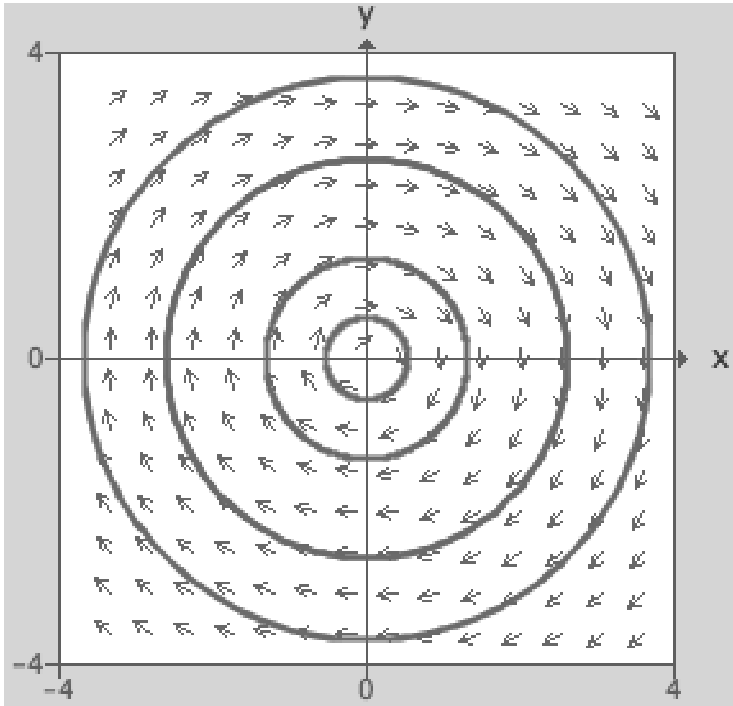
$$Y_{imag}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix},$$

are both solutions of this differential equation. And notice that the real part starts out at the point (1, 0) whereas the imaginary part begins at (0, −1). So these are independent solutions. It is interesting that using just one of the eigenvalues gives us a pair of solutions that then generate all possible solutions (below) exactly as we saw earlier.

$$Y(t) = k_1\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + k_2\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

Clearly, when $k_1 = 0$, our solutions all lie on circles centered at the origin. In fact, that's true for any values of k_1 and k_2 (as long as they are both not equal to zero). So our phase plane here is seen in Figure 13.1.

Figure 13.1



This type of equilibrium is center. In general, if the eigenvalues are $\pm ia$, the solutions may lie on circles or ellipses.

When we compute the roots of a characteristic equation, we may also find complex roots of the form $a + ib$ and $a - ib$ where, unlike the previous case, a may be nonzero. In this case, the exact same procedure as above yields a complex eigenvector V for one of the eigenvalues. This then yields, by Euler's formula, solutions of the form

$$e^{(a+ib)t}V = e^{at}(\cos(bt) + i\sin(bt))\,V\,.$$

If we then break this vector solution into its real and imaginary parts, we find 2 real solutions to the linear system, and it turns out that these solutions are independent. So we find the general solution just as before.

The difference here is that our new solutions always involve the exponential term $e^{(at)}$. If $a > 0$, then this term tends to infinity as time goes on, whereas if $a < 0$, this term goes to zero as time goes on. As the remaining terms in the solution involve $\sin(bt)$ and $\cos(bt)$, it follows that if $a > 0$, solutions

spiral away from the origin (the equilibrium is a **spiral source**), but if $a < 0$, solutions spiral in toward the origin (a spiral sink).

For example, for the linear system

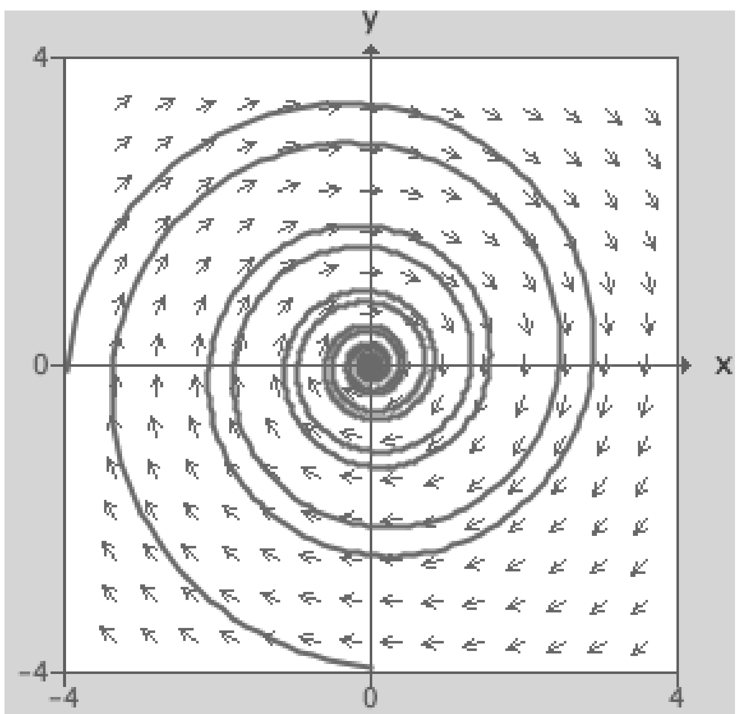
$$Y' = \begin{pmatrix} -0.1 & 1 \\ -1 & -0.1 \end{pmatrix} Y,$$

we can easily check that the eigenvalues are $-0.1 + i$ and $-0.1 - i$. Similar computations to the above then yield the general solution

$$Y(t) = k_1 e^{-0.1t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + k_2 e^{-0.1t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

The corresponding phase plane is the below.

Figure 13.2



If we consider instead the linear system

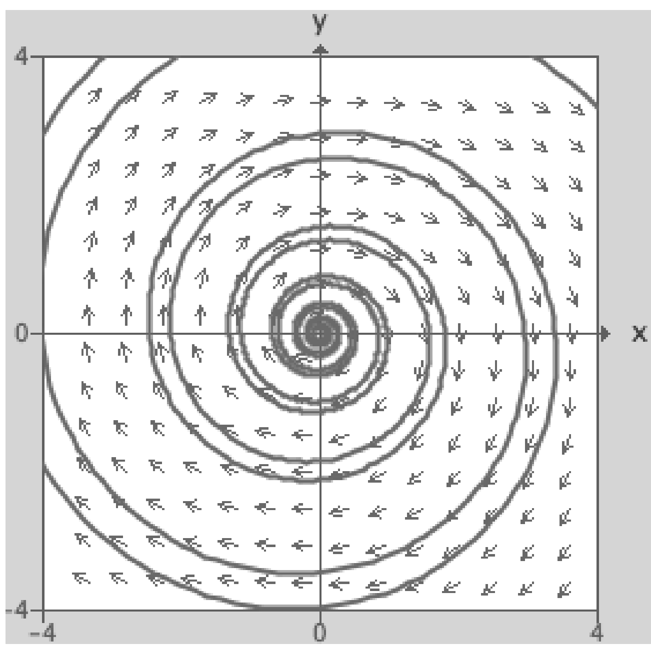
$$Y' = \begin{pmatrix} 0.1 & 1 \\ -1 & 0.1 \end{pmatrix} Y ,$$

the eigenvalues are $0.1 + i$ and $0.1 - i$, and the general solution becomes

$$Y(t) = k_1 e^{0.1t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + k_2 e^{0.1t} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} .$$

Solutions now spiral away from the origin, as seen below.

Figure 13.3



So far we have figured out how to handle the major types of linear systems, sinks, saddles, sources, and centers. But there are some other special cases, such as when we have zero eigenvalues and real and repeated eigenvalues.

If we have just one eigenvalue that is equal to zero (and so the other is either positive or negative), we know more or less how to solve the system. We find our eigenvector V corresponding to the eigenvalue 0, so we then have a solution of the form $Y(t) = k_1 e^{0t} V$. Our solution is just the constant

$k_1 V$. So all points that lie on the straight line passing through the origin and the point V in the plane are constant or equilibrium solutions. Then the sign of the other eigenvalue a determines whether the remaining solutions tend toward or away from these equilibria.

As an example, consider the system

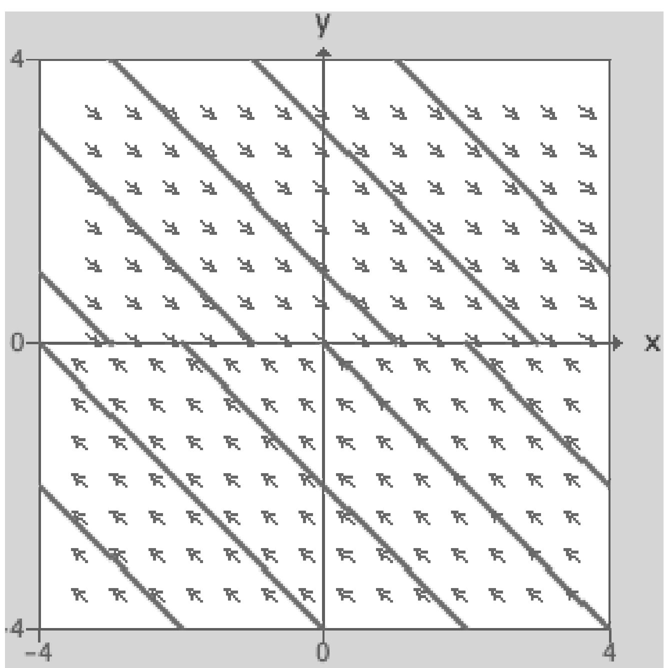
$$Y' = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} Y.$$

Check on your own that the eigenvalues are 0 and −1 and the general solution is

$$Y(t) = k_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So we have a line of equilibrium points along the x-axis, and all other solutions tend toward these equilibria. The phase plane is the below.

Figure 13.4



As we'll see in the next lecture when we describe the trace-determinant plane, linear systems often undergo bifurcations when there is a family of such systems that has either a zero eigenvalue or a center. For example, consider the family of differential equations

$$Y' = \begin{pmatrix} -1 & 0 \\ 1 & A \end{pmatrix} Y$$

The eigenvalues here are -1 and A. So, whenever $A < 0$, we have a sink; whenever $A > 0$, we have a saddle. A bifurcation occurs when $A = 0$, and the system has a zero eigenvalue.

Important Term

spiral source: An equilibrium solution of a system of differential equations for which all nearby solutions spiral away from it.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 3.2–3.3.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 3.1.

Kolman and Hill, *Elementary Linear Algebra*, chap. 7.

Lay, *Linear Algebra*, chap. 5.

Roberts, *Ordinary Differential Equations*, chap. 7.

Strang, *Linear Algebra and Its Applications*, chap. 5.

Strogatz, *Nonlinear Dynamics and Chaos*, chaps. 5.1–5.2.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Linear Phase Portraits.

Problems

1. **a.** For the system of differential equations

$$Y' = \begin{pmatrix} 2 & 5 \\ -1 & -2 \end{pmatrix} Y,$$

first compute the characteristic equation.

b. What are the eigenvalues of this matrix?

c. Compute the asssociated eigenvectors.

2. **a.** Compute the eigenvalues associated to

$$Y' = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} Y.$$

b. Find the associated eigenvectors for this system.

3. Rewrite the second-order differential equation for the mass-spring system as a linear system and find the eigenvalues. Do you see any connection with our earlier method of solving this system?

4. **a.** What are the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}?$$

b. Determine the types of the equilibrium points for the linear system corresponding to the above matrix, and sketch the regions in the *a*-*b* plane where this matrix has different types of eigenvalues.

5. **a.** Find the general solution of the system

$$Y' = \begin{pmatrix} 1 & 3 \\ \sqrt{2} & 3\sqrt{2} \end{pmatrix} Y$$

and sketch the phase plane.

b. Find the solution to this system satisfying the initial condition $Y(0) = (1, 0)$.

Exploration

Consider all possible 2×2 matrices of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Determine the set of a-, b-, c-, and d-values for which this matrix has either a zero eigenvalue or pure imaginary eigenvalues (i.e., eigenvalues of the form $\pm\mu$ for some $\mu \neq 0$). Express these sets in terms of the trace and the determinant of A.

Summarizing All Possible Linear Solutions

Lecture 14

We have solved almost every type of linear system. The only remaining case is that of repeated eigenvalues. The easy case in this situation is when our system assumes the form

$$Y' = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} Y .$$

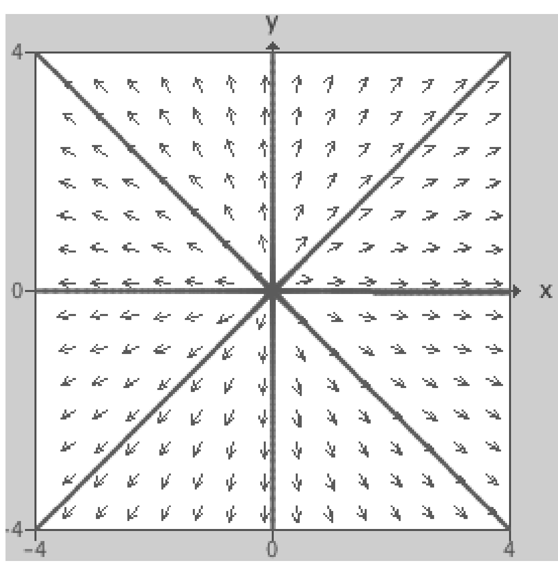
We see immediately that the eigenvalues are both given by a, and any nonzero vector is an eigenvector since

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = a \begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore all solutions are straight line solutions (assuming a is nonzero) either tending to the origin (if $a < 0$) or tending away (if $a > 0$).

Here is the phase plane when $a = 1$.

Figure 14.1



This case, however, is unusual. Most often we find that the eigenvectors lie along a single straight line. To find the other solutions, consider the system given by

$$Y' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} Y .$$

It is easy to check that the eigenvalues here are −1 (repeated), and there is a single eigenvector given by (1, −1). So we have one straight line solution given by

$$Y_1(t) = k_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

To get the other solutions, recall that we have actually seen this example before; it is the linear system corresponding to the mass-spring system determined by the equation $y'' + 2y' + y = 0$. We saw earlier that this was the critically damped harmonic oscillator whose general solution was given by $y(t) = k_1 e^{-t} + k_2 te^{-t}$. As a system, our solution is then given by

$$Y(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} k_1 e^{-t} + k_2 te^{-t} \\ -k_1 e^{-t} - k_2 te^{-t} + k_2 e^{-t} \end{pmatrix},$$

which we can rewrite as

$$Y(t) = k_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 te^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Note that the eigenvector (1, −1) appears twice in this expression. But there is another term involving

$$k_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

We easily check that the vector (0, 1) is actually the solution of the equation

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where λ is our known repeated eigenvalue. In fact, this is the prescription for finding the general solution for a linear system with repeated eigenvalue λ and a single eigenvector. So the general solution is

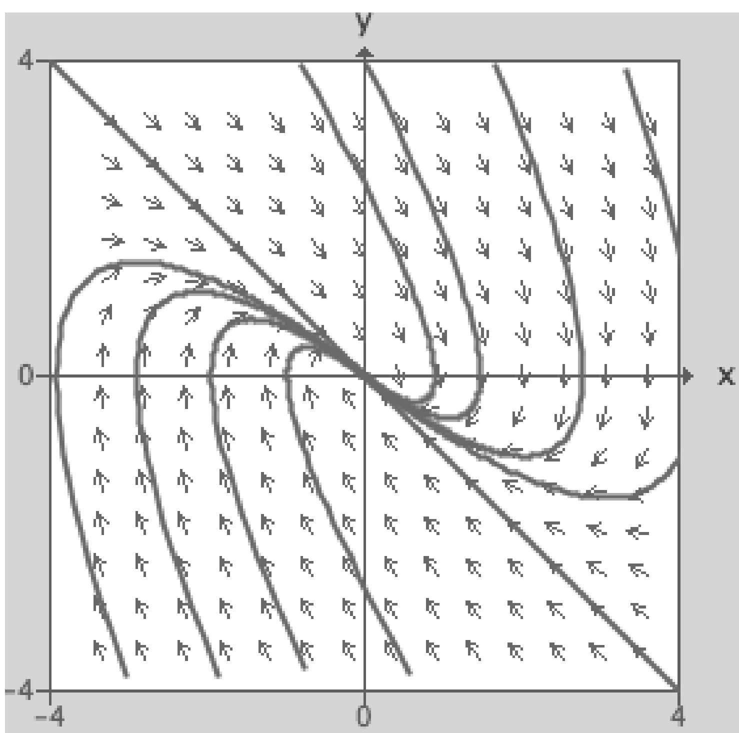
$$Y(t) = k_1 e^{\lambda t}\ V + k_2 t\, e^{\lambda t}\ V + k_2 e^{\lambda t}\ W$$

where we have

$$(A - \lambda I)W = V\ .$$

The phase plane for the above system now has only one straight line solution, while all other solutions also tend to the origin since the terms e^{-t} and te^{-t} both tend to zero as time goes on.

Figure 14.2



Indeed, as seen above, all other solutions tend to the origin in a direction tangential to the straight line solutions. The reason for this is that each of the 3 terms in the solution involves either e^{-t} or te^{-t}. By calculus, as t tends to infinity, e^{-t} tends to zero much more quickly than te^{-t}, forcing all solutions to come into the origin tangent to the line containing the eigenvector.

Now let's summarize the entire situation for linear systems by painting the picture of the trace-determinant plane. For a linear system of the form

$$Y' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y ,$$

we have seen that the characteristic equation is given by

$$\lambda^2 - T\lambda + D\lambda = 0 .$$

Here the trace T is given by $a + d$, and the determinant D is $ad - bc$. Then the eigenvalues are given by

$$\frac{T \pm \sqrt{T^2 - 4D}}{2} .$$

Using this formula, we can summarize the situation for linear systems in writing as follows.

1. The system has complex eigenvalues if $T^2 - 4D < 0$. So we have

- **a.** a spiral sink if $T < 0$,
- **b.** a spiral source if $T > 0$, and
- **c.** a center if $T = 0$.

2. The system has real, distinct eigenvalues if $T^2 - 4D > 0$. So we have

   a. a saddle if $D < 0$,

   b. a real sink if $D > 0$ and $T > 0$, and

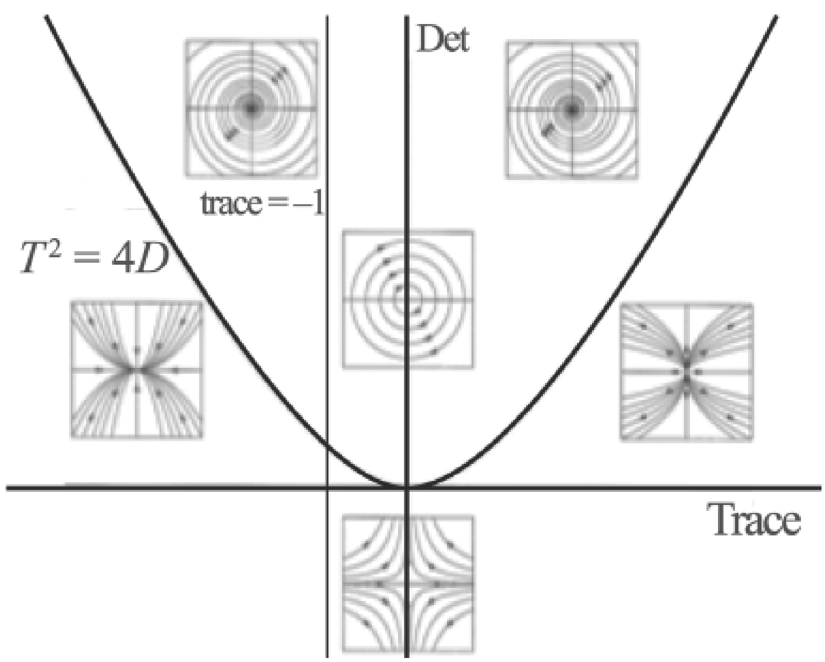
   c. a real source if $D > 0$ and $T < 0$.

3. The system has a single zero eigenvalue if $D = 0$ and $T \neq 0$.

4. The system has repeated eigenvalues if $T^2 - 4D = 0$.

We can also summarize this pictorially by drawing the trace-determinant plane. In this figure, the horizontal axis is the trace axis and the vertical axis is the determinant axis. Each matrix then corresponds to a point in this plane given by (T, D). In the figure below, the different regions then correspond to places where our linear system has different phase planes. The parabola given by $T^2 - 4D = 0$ represents the matrices that have repeated eigenvalues; the trace axis $D = 0$ is where we have a zero eigenvalue; and the positive determinant axis is where we have a center. The regions between these 3 curves are where the other types of phase planes occur.

Figure 14.3



This figure also represents the bifurcation diagram for linear systems. For if we have a family of linear systems that depends on a parameter k, then the corresponding curve that this family produces in the trace-determinant plane shows how and when bifurcations occur. For example, consider the family of linear systems given by

$$Y' = \begin{pmatrix} 0 & k \\ -1 & -1 \end{pmatrix} Y \,.$$

The trace here is −1, and the determinant is k. So this family of systems lies along the vertical line given by trace = −1 and the determinant increasing. We see that this family starts out as a saddle, and then crosses the zero eigenvalue line and becomes a real sink. We eventually find repeated eigenvalues followed by a spiral sink.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps 3.5 and 3.7.

Hirsch, Smale, and Devaney, *Differential Equations*, chaps. 3.3 and 4.1.

Strang, *Linear Algebra and Its Applications*, chap. 5.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 5.2.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Linear Phase Portraits.

Problems

1. **a.** Sketch the direction field for the system

$$Y' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} Y\ .$$

b. What are the eigenvalues for this matrix?

c. Find all possible eigenvectors for this matrix.

d. What is the general solution of this system?

e. How do typical solutions of this system behave?

2. Find the general solution of the linear system

$$Y' = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} Y$$

and sketch the phase plane.

3. Find the general solution of the linear system

$$Y' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Y$$

and sketch the phase plane.

4. **a.** Sketch the curve in the trace-determinant plane that the following family of linear systems moves along as a changes.

$$Y' = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix} Y$$

b. Find all a-values where this system undergoes a bifurcation (i.e., the type of the equilibrium point at the origin changes).

5. Repeat problems 4a and 4b for the system

$$Y' = \begin{pmatrix} a & 2 \\ -2 & 0 \end{pmatrix} Y .$$

Exploration

Consider the 3-parameter family of linear systems given by

$$Y' = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} Y .$$

First fix $a > 0$ and describe the analogue of the trace determinant plane (i.e., sketch the regions in the b-c planes where this system has different types of equilibria. Then repeat this question for $a = 0$ and $a < 0$. Put all of this information together to create a 3-dimensional model that describes all possible behaviors of this system.

Nonlinear Systems Viewed Globally—Nullclines

Lecture 15

We now move on to nonlinear systems of ordinary differential equations. In this lecture we introduce one of the main techniques that we use to understand the behavior of solutions of nonlinear systems. This tool, called nullclines, allows us to partition the phase plane into certain regions where we know the approximate direction of the vector field.

Given a nonlinear system $x' = F(x, y)$ and $y' = G(x, y)$, the x-nullcline will be the set of points at which $x' = 0$—that is, $F(x, y) = 0$. At these points, the vector field points vertically, either up or down. Similarly, the y-nullcline is the set of points where $y' = 0$, so $G(x, y)$ vanishes. At these points, the vector field points horizontally, either to the left or the right. Note that the equilibrium points are then given by the points of intersection of the x- and y-nullclines.

Most importantly, in the regions between the nullclines, assuming our vector field is continuous, the vector field must point in 1 of 4 directions: northeast, northwest, southeast, or southwest. This very often allows us to get a good handle on the qualitative behavior of nonlinear systems.

Let's start with a simple nonlinear system.

$$x' = y - x^2$$

$$y' = 1 - y$$

The x-nullcline is the parabola $y = x^2$. The vector field is vertical along this parabola. It points to the right above the parabola since $x' > 0$ in this region, and it points to the left below the parabola. The y-nullcline is the horizontal line $y = 1$, so we immediately see equilibrium points at $(-1, 1)$ and $(1, 1)$. Moreover, the vector field is tangent to this nullcline. So we have solutions that move right along this line in the region above the parabola and to the

left on the portions below the parabola. We also have that $y' < 0$ above the line $y = 1$ and that $y' > 0$ below this line. Therefore we know the direction of solutions in all the regions between the nullclines.

If a solution starts in the region above the parabola but below the line $y = 1$, it must proceed up and to the right. It cannot cross the line $y = 1$, since we have a solution there and solutions cannot cross by the **existence and uniqueness theorem**. It also cannot hit the parabola because to do so, it must turn and become vertical. But the vector field always points upward here, so this is not possible. So solutions in this region must tend to the equilibrium point at (1, 1). The same thing happens in the region to the right of the parabola and above y =1; all these solutions tend to the same equilibrium point. In the left region below the parabola but above $y = 1$, solutions must tend to infinity.

In the region above the parabola and $y = 1$, solutions have 4 choices: They can tend directly to (1, 1). They can cross the right branch of the parabola but then enter the region where all solutions tend to (1, 0). They can hit the left branch of the parabola, in which case they tend to infinity. Or they can tend to the equilibrium point at (−1, 1). In the lower region, solutions similarly have 4 possible choices, but we know more or less what happens to all solutions. It appears that (1, 1) is a sink and (−1, 1) is a saddle. But the latter is not clear: Can we have more than a single curve tending to that equilibrium point? We'll see how to handle this in the next lecture.

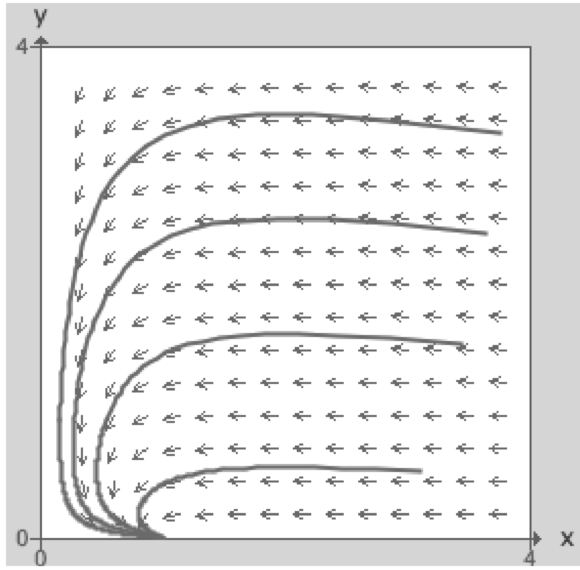
Let's go back to the predator-prey system

$$x' = x(1 - x) - xy$$

$$y' = -y + axy.$$

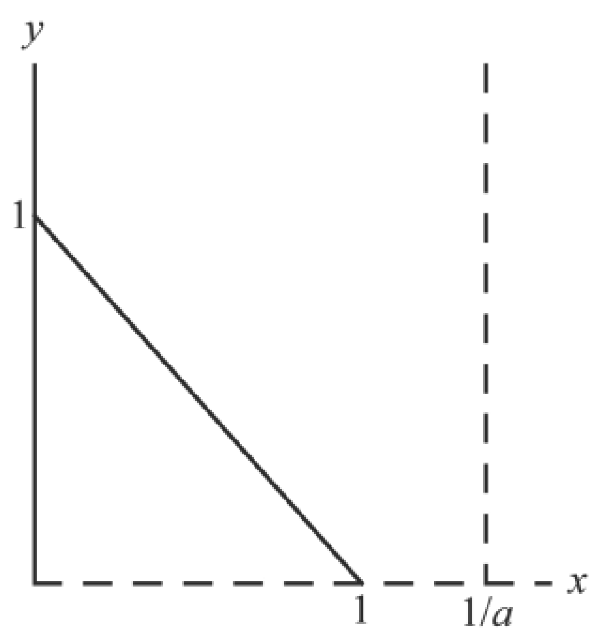
Here we first choose $0 < a < 1$, so the rate of predation is small. Recall that the computer showed us that all solutions tended to the equilibrium point at $x = 1, y = 0$.

Figure 15.1



The x-nullcline will be the set of points in the phase plane where the x-component of the vector field vanishes. So the x-nullcline is the set of points where the vector field points vertically. In this example, the x-nullcline is given by $x(1 - x) - xy = 0$, so that $xy = x(1 - x)$. Thus the x-nullcline consists of the pair of straight lines $x = 0$ and $y = 1 - x$. Similarly, the y-nullcline is the set of points where $y' = 0$ (i.e., where the vector field points horizontally). In this case, the y-nullclines are the lines $y = 0$ and $x = 1/a$.

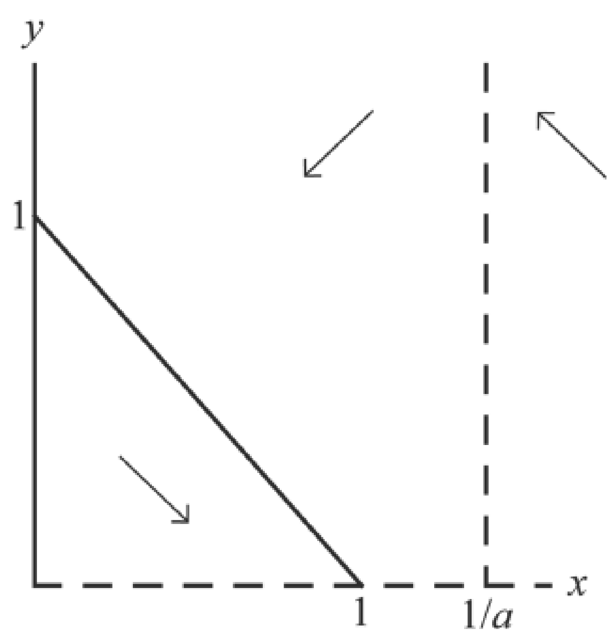
Figure 15.2



In this figure, the solid lines are the x-nullclines and the y-nullclines are dashed. As before, the places where an x- and a y-nullcline intersect are the equilibrium points for the system. So in this case, the equilibria are (0, 0) and (1, 0). Note also that the y-nullcline $x = 1/a$ and the x-nullcline $y = 1 - x$ meet at a point where the y-coordinate is negative. This means that this is not an equilibrium point for the predator-prey system, since both x and y are nonnegative.

Most important, at all points in one of the regions between the nullclines, the vector field always points in 1 of 4 directions: northeast, northwest, southeast, or southwest. And we can determine this direction by simply computing the vector field at a single point in the region. For example, at (1, 1), our vector field is given by $x' = -1$, $y' = -1 + a < 0$; so the vector field points down and to the left at all points between the lines $y = 1 - x$ and $x = 1/a$. In the triangle bounded by $y = 1 - x$ and the x- and y-axes, the vector field points down and to the right. And in the region to the right of $x = 1/a$, the vector field points up and to the left.

Figure 15.3

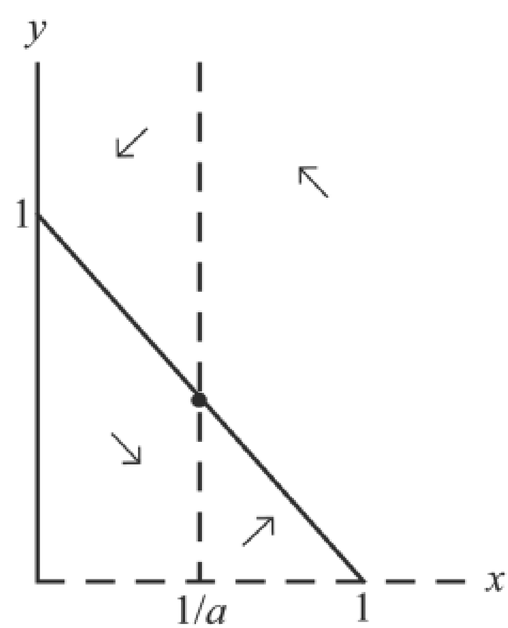


As a consequence, we completely understand what happens to all solutions of this system. Solutions that start to the right of $x = 1/a$ must move up and to the right until they hit the line $x = 1/a$. Then they move down and to the left. So there are only 2 possibilities for what happens to these solutions: Either the solution tends to the equilibrium point at (1, 0) or it crosses the line $y = 1 - x$. (They can never cross the x- or y-axes by the uniqueness theorem.)

In the final case, in the triangular region, solutions move down and to the right. Since these solutions can never cross the x-axis, they must also tend to the equilibrium point at (1, 0). We therefore know that all solutions of this predator-prey system necessarily tend to the equilibrium point at (1, 0), assuming that our initial populations are both positive. Therefore we know that when $a < 1$, the predator population goes extinct.

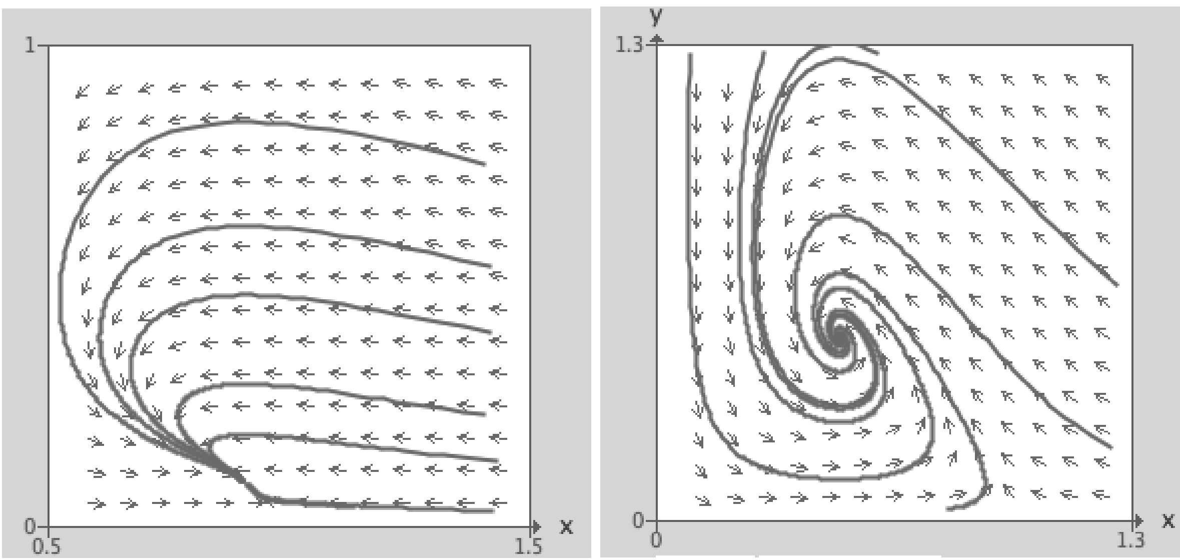
Nullclines do not always give the complete picture of the phase plane. For example, if $a > 1$ in our predator-prey system, we get the following nullcline picture. Here the nullclines $x = 1/a$ and $y = 1 - x$ do cross at a new equilibrium point $(1/a, (a - 1)/a)$.

Figure 15.4



We do know that every solution starting at a point with x, $y > 0$ must eventually enter the region $x \leq 1/a$. But does this solution tend to the new equilibrium point? If so, does the solution spiral into this equilibrium point, or does it not spiral? The computer seems to indicate that both types of behavior can happen. These figures below show the phase plane when $a = 1.2$ (left) and when $a = 2$ (right).

Figure 15.5



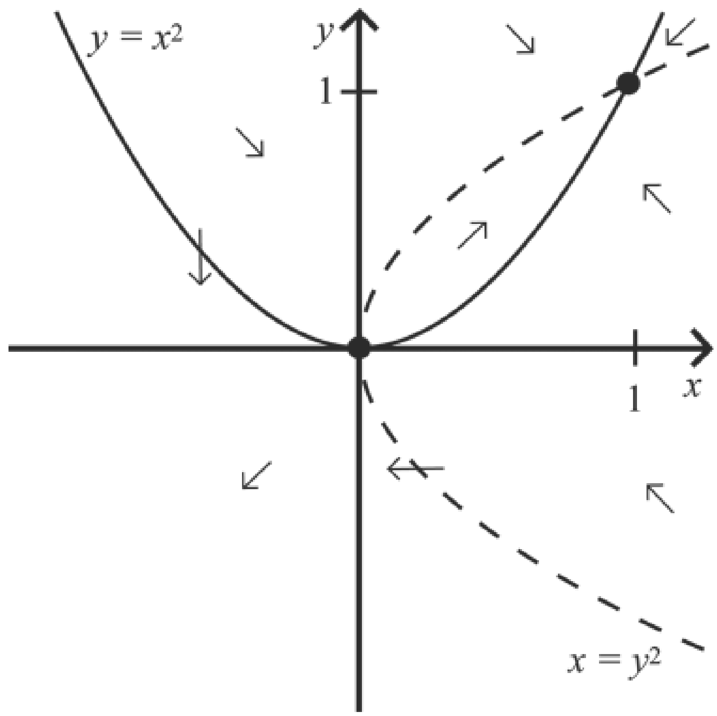
In addition, could the solution move around the equilibrium point in a periodic motion and never come to rest? This kind of local behavior near equilibria will be the subject of the next lecture.

As another example of nullcline analysis, consider the nonlinear system

$$y' = x - y^2.$$

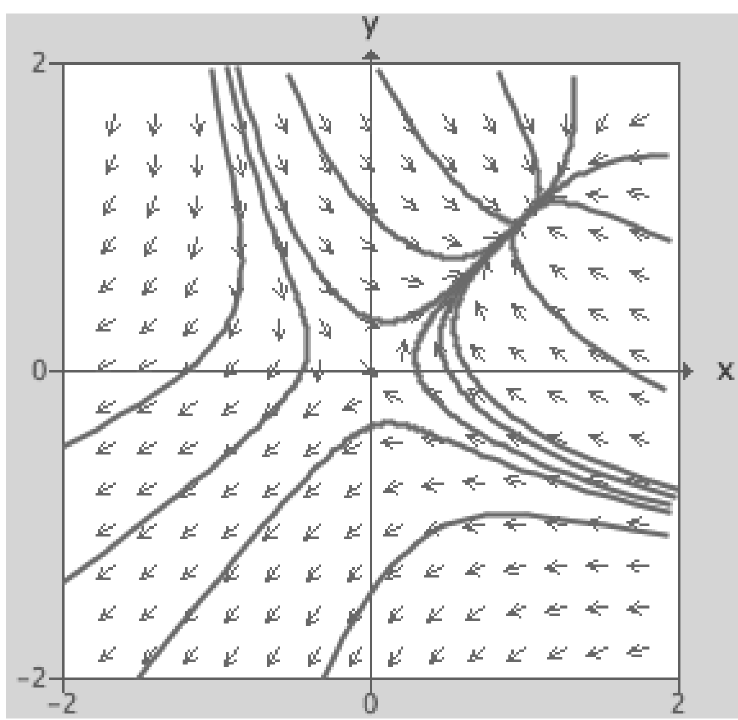
The x-nullcline for this system is the parabola $y = x^2$, which opens upward; the y-nullcline is the parabola $x = y^2$, which opens to the right. These nullclines meet at the 2 equilibria: (0, 0) and (1, 1). In the 5 regions separated by the nullclines, we compute that the vector field points as follows.

Figure 15.6



So (1, 1) appears to be a sink and (0, 0) a saddle. We don't know this for sure, but the next lecture will give us the local tool to make this assessment. The actual phase plane for this system is below.

Figure 15.7



Important Term

existence and uniqueness theorem: This theorem says that, if the right-hand side of the differential equation is nice (basically, it is continuously differentiable in all variables), then we know we have a solution to that equation that passes through any given initial value, and, more importantly, that solution is unique. We cannot have two solutions that pass through the same point. This theorem also holds for systems of differential equations whenever the right side for all of those equations is nice.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 5.2.

Edelstein-Keshet, *Mathematical Models in Biology*, chap. 6.2.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 9.1.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 6.1.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Linear Phase Portraits.

Problems

1. **a.** Find the x- and y-nullclines for the system

$$x' = y$$

$$y' = 2x.$$

b. Sketch the regions where the vector field points in different directions.

c. What are the equilibrium points for this system?

d. Sketch the direction field and phase plane for this system.

e. How many solutions tend to the equilibrium points?

2. Sketch the nullclines and phase plane in the region x, $y \geq 0$ for the following system.

$$x' = x(-x - 3y + 150)$$
$$y' = y(-2x - y + 100)$$

3. Sketch the nullclines and phase plane in the region x, $y \geq 0$ for the following system.

$$x' = x(100 - x - 2y)$$
$$y' = y(150 - x - 6y)$$

4. Using nullclines, determine the phase plane for the following system.

$$x' = x(1 - x)$$
$$y' = x - y^2$$

5. **a.** Consider the predator-prey system

$$x' = x(1 - x) - Axy$$

$$y' = y(1 - y) + xy,$$

where A is a positive parameter. Using nullclines, determine the values of A where different outcomes occur. In particular, determine at which parameters this system undergoes a bifurcation.

b. What changes will there be in the predator-prey system if we introduce a new parameter of B instead of A?

$$x' = x(1 - x) - xy$$

$$y' = y(1 - y) + Bxy$$

Exploration

Use the nullclines to investigate the phase plane for the following system that depends on a parameter A.

$$x' = x^2 - 1$$

$$y' = -xy + A(x^2 - 1)$$

At what values of A do you see a bifurcation? Explain what happens to solutions before and after the bifurcation.

Nonlinear Systems near Equilibria—Linearization

Lecture 16

Recall that for the predator-prey system in the last lecture

$$x' = x(1 - x) - xy$$

$$y' = -y + axy$$

we had an equilibrium point at $x = 1/a$, $y = (a - 1)/a$ when $a > 1$. It appeared from the computer illustrations that sometimes solutions spiraled in to this point, but for other a-values this did not occur. To understand better what happens near equilibria in nonlinear systems, we use a procedure called linearization. This often allows us to use a linear system to figure out what is happening locally near a nonlinear equilibrium point.

To explain the process of linearization, let's first consider a first-order equation $y' = f(y)$, and let's assume we have an equilibrium point at $y = 0$. Recall from calculus that we can expand the function $f(y)$ as a **Taylor series** about $y = 0$. That is, we can write it as follows.

$$f(y) = f(0) + f'(0)\,y + \frac{f''(0)}{2!}y^2 + \frac{f'''(0)}{3!}y^3 + \cdots$$

The first term in this series is 0, since we have an equilibrium point at $y = 0$. Also, if y is very close to 0, the y^2 term (the third term in the series) is much smaller than the term involving just y (the second term). Similarly, the terms involving y^n for $n > 2$ are even smaller. This tells us that we can approximate the right-hand side of $y' = f(y)$ by just using the second term in the Taylor series, namely $f'(0)y$. Therefore, very close to $y = 0$, solutions of our nonlinear ODE $y' = f(y)$ should be close to solutions of the ODE $y' = f'(0)y$, which is just a linear ODE that we know how to solve.

If our equilibrium point is located at a nonzero value, say y_0, then a simple change of variables says that the same process is true: Solutions of the

nonlinear ODE resemble the solutions of the linear equation $y' = f'(y_0)y$, at least in a very small neighborhood of $y = y_0$. There is one slight problem here. If $f'(y_0)$ happens to equal 0, then the linear system reduces to just $y' = 0$, for which all solutions are equilibria. This certainly will not be the case for the corresponding nonlinear system.

For a nonlinear system

$$x' = F(x, y)$$

$$y' = G(x, y)$$

with an equilibrium point at (x_0, y_o), we can perform a similar linearization, only now the corresponding linear approximation involves the partial derivatives of both F and G. That is, our nonlinear system has solutions near (x_0, y_o) that resemble the solutions of the following linear system.

$$x' = \frac{\partial F}{\partial x}(x_0, y_0)\,x + \frac{\partial F}{\partial y}(x_0, y_0)\,y$$

$$y' = \frac{\partial G}{\partial x}(x_0, y_0)\,x + \frac{\partial G}{\partial y}(x_0, y_0)\,y$$

Note that this is a linear system of ODEs. Again, what we have done here is just drop all the higher-order terms in the Taylor series expansions of both F and G. As above, solutions of our nonlinear system near the equilibrium point (x_0, y_o) should resemble the solutions of the linear system

$$Y' = J\,Y\,,$$

where J is the matrix

$$\begin{pmatrix} \dfrac{\partial F}{\partial x}(x_0, y_0) & \dfrac{\partial F}{\partial y}(x_0, y_0) \\ \dfrac{\partial G}{\partial x}(x_0, y_0) & \dfrac{\partial G}{\partial y}(x_0, y_0) \end{pmatrix}.$$

The matrix J is called the **Jacobian matrix** for the nonlinear system.

As an example, consider the system below.

$$x' = x - xy^2$$

$$y' = -y + xy$$

There is no way to solve this system explicitly, but we certainly have an equilibrium point at the origin. The Jacobian matrix at an arbitrary point is

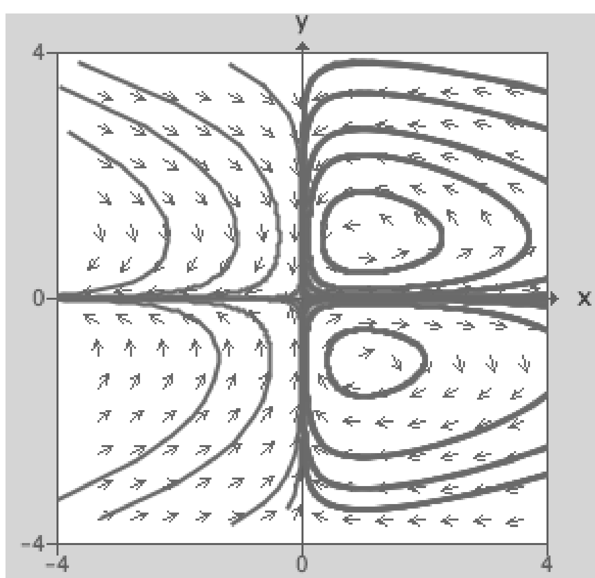
$$J = \begin{pmatrix} 1-y^2 & -2xy \\ y & -1+x \end{pmatrix}.$$

At the origin we get

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

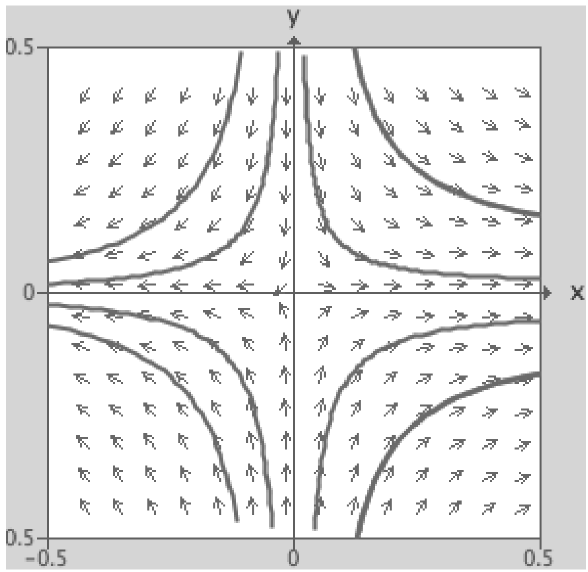
so our nonlinear system is close to $Y' = JY$. Since J has eigenvalues 1 and -1, our nonlinear system has a saddle at the origin. A glance at the phase plane shows that there is a lot more going on in this system.

Figure 16.1



However, if we zoom in at the origin, we do see a part of the phase plane that looks like a saddle.

Figure 16.2



As in the first-order case, there are times when linearization does not work. In particular, if the Jacobian matrix associated to our equilibrium point has imaginary or zero eigenvalues, then the addition of the higher-order terms will usually greatly alter the linear phase plane. For example, the system

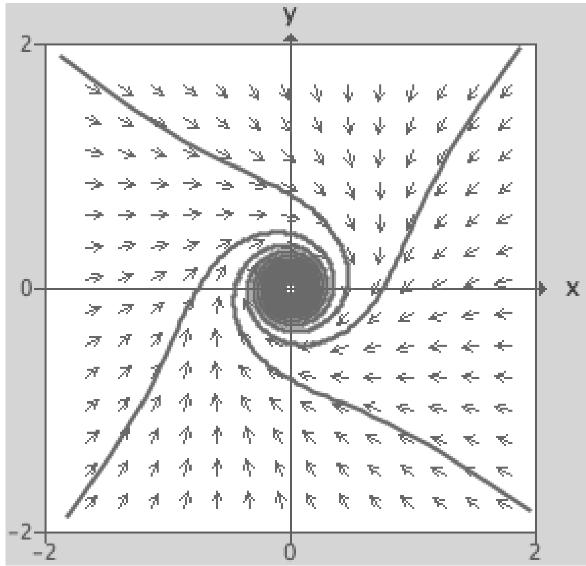
$$x' = y - x(x^2 + y^2)$$
$$y' = -x - y(x^2 + y^2)$$

has an equilibrium point at the origin. The corresponding Jacobian matrix is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the eigenvalues are easily seen to be the imaginary numbers i and $-i$. So the linearized system has a center at the origin. But this is not true of the nonlinear system (Figure 16.3), where we see a spiral sink.

Figure 16.3



If we look instead at the system

$$x' = y + x(x^2 + y^2)$$
$$y' = -x + y(x^2 + y^2)$$

then again the linearized system has a center at the origin. But the same arguments as above show that this equilibrium point is now a source.

Let's return to the predator-prey system

$$x' = x(1 - x) - xy = F(x, y)$$
$$y' = -y + axy = G(x, y).$$

As we saw earlier, we have an equilibrium point at $(1/a, (a - 1)/a)$ when $a > 1$. The Jacobian matrix at an arbitrary point is

$$J = \begin{pmatrix} 1-2x+y & -x \\ ay & -1+ax \end{pmatrix}.$$

At our equilibrium point, this matrix becomes

$$J = \begin{pmatrix} -1/a & -1/a \\ a-1 & 0 \end{pmatrix},$$

so the solutions near our equilibrium point should resemble those of the linearized system $Y' = JY$. The characteristic equation for the linearized system is

$$\lambda^2 + \lambda / a + (a-1)/a = 0,$$

so the eigenvalues are

$$\tfrac{1}{2a}\left(-1 \pm \sqrt{1+4a(1-a)}\right).$$

There are then 2 different cases. The first occurs when the term inside the square root is negative. In this case, the eigenvalues are complex with negative real parts, so the equilibrium is a spiral sink. If the term $1 + 4a(1 - a) > 0$, we certainly know that this term is less than 1. This is because $a > 1$, which makes $4a(1 - a)$ negative. Our eigenvalues are both real and negative, so in this case we have a real sink. An easy computation shows that the first case occurs when $a > 2 + 2\sqrt{2}$, and the second case occurs if $1 < a < 2 + 2\sqrt{2}$. This is what we saw in the previous lecture.

Important Terms

Jacobian matrix: A matrix made up of the various partial derivatives of the right-hand side of a system evaluated at an equilibrium point. The corresponding linear system given by this matrix often has solutions that resemble the solutions of the nonlinear system, at least close to the equilibrium point.

Taylor series: Method of expanding a function into an infinite series of increasingly higher-order derivatives of that function. This is used, for example, when we approximate a differential equation via the technique of linearization.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 5.1.

Edelstein-Keshet, *Mathematical Models in Biology*, chaps. 4.7–4.8.

Hirsch, Smale, and Devaney, *Differential Equations*, chaps. 8.1–8.3.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 6.3.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, HPG Linearizer, HPG System Solver, Vander Pol.

Problems

1. Compute the partial derivatives with respect to x and y of the function $F(x, y) = (x^2y + x^3 + y^3)$.

2. What is the Jacobian matrix for a linear system

$$x' = ax + by$$

$$y' = cx + dy?$$

3. **a.** Find all equilibrium points for the system

$$x' = x^2 + y$$
$$y' = x.$$

b. Compute the Jacobian matrix at each equilibrium point.

c. What is the type of these equilibrium points?

4. Find and determine the type of the equilibrium points for the Vander Pol equation given by

$$x' = y$$
$$y' = -x + (1 - x^2)y,$$

and then use the computer to see what else is going on for this system.

5. Give an example of a nonlinear system that has an equilibrium point for which the linearized system has a zero eigenvalue, but the nonlinear system has only one equilibrium point.

6. For the system

$$x' = y - 1$$
$$y' = y - x^2,$$

determine the types of each equilibrium point.

7. For the system

$$x' = \sin(x)$$
$$y' = \cos(y),$$

find all equilibrium points and use linearization together with the nullclines to sketch the phase plane.

8. For the system

$$x' = y - x^2 + A$$
$$y' = y,$$

determine the types of each equilibrium point and also the A-values where a bifurcation occurs.

Exploration

Consider the system of differential equations

$$x' = x/2 - y - (x^3 + y^2x)/2$$
$$y' = x + y/2 - (y^3 + x^2y)/2.$$

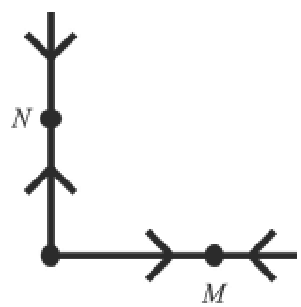
What is the type of the equilibrium point at the origin? Using the computer, find out what happens far away from the origin. To understand this behavior, change coordinates to polar coordinates. Then you will be able to solve this system explicitly.

Bifurcations in a Competing Species Model

Lecture 17

Let's combine the previous topics to investigate what happens when we have 2 species that compete with each other for the same resources. Let x and y denote the population of these 2 species. We first assume that if the other species is absent, the population of the given species obeys the limited-growth population model. So when $y = 0$, we will assume that the x-population is given by $x' = x(1 - x/M)$. When $x = 0$, we have $y' = y(1 - y/N)$. So our phase plane thus far is the below.

Figure 17.1



Our second assumption is that if the y population increases, it should have a negative effect on the x-population, and vice versa. So if y increases, x' should decrease. One way to model this is by adding the term $-(a/M)xy$ to the differential equation for x and $-(b/N)xy$ to the y-equation. So our competing species system is the below.

$$\frac{dx}{dt} = x(1 - x/M) - (a/M)xy$$

$$\frac{dy}{dt} = y(1 - y/N) - (b/N)xy$$

We have 4 parameters for this system: a, b, M, and N. The quantities a and b measure how competitive the 2 species are. For simplicity, let us assume that $M = N = 400$; the equations are below. (Note: There is nothing special

about the number 400; it is simply the default value of M and N in the software we use.)

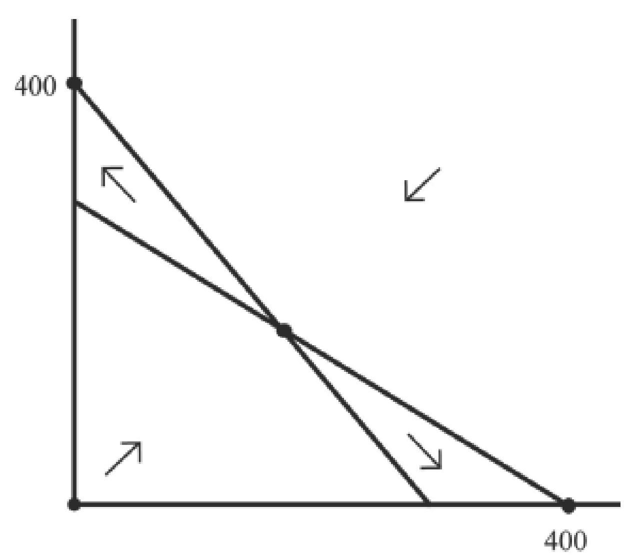
$$\frac{dx}{dt} = x(1 - x/400 - ay/400)$$

$$\frac{dy}{dt} = y(1 - y/400 - bx/400)$$

Now let's view the phase plane and nullclines. First assume that a, $b > 1$ (so our species are very competitive). The x-nullclines lie on the y-axis (where we know what happens) and along the straight line $1 - (x/400) - ay/400 = 0$, or $y = 400/a - x/a$. Similarly, the y-nullclines are given by the x-axis and the line $y = 400 - bx$. Note that the 2 nullclines that do not lie on the axes meet at the point $x = 400(1 - a)/(1 - ab)$, $y = 400(1- b(1 - a)/(1 - ab))$. This is another equilibrium point for our system since a, $b > 0$. We call this the coexistence equilibrium point.

At the point (400, 400), we compute that $x' = -400a$ and $y' = -400b$, so the vector field points down and to the left. Similarly, at (1, 1) we have $x' > 0$ and $y' > 0$, so the vector field points up and to the right. We similarly compute the value of the vector field in the other regions bounded by the nullclines to obtain the following figure.

Figure 17.2



It appears that the coexistence equilibrium point is a saddle. Of course, we do not know this for sure from the picture. However, linearization at this point should give us the answer. For simplicity, assume $a = b = 2$; so our equations are

$$x' = x - x^2/400 - xy/200$$

$$y' = y - y^2/400 - xy/200.$$

The coexistence equilibrium point is found by solving the 2 equations

$$1 - x/400 - y/200 = 0$$

$$1 - y/400 - x/200 = 0$$

simultaneously. A little algebra yields $x = y = 400/3$. Then the Jacobian matrix is

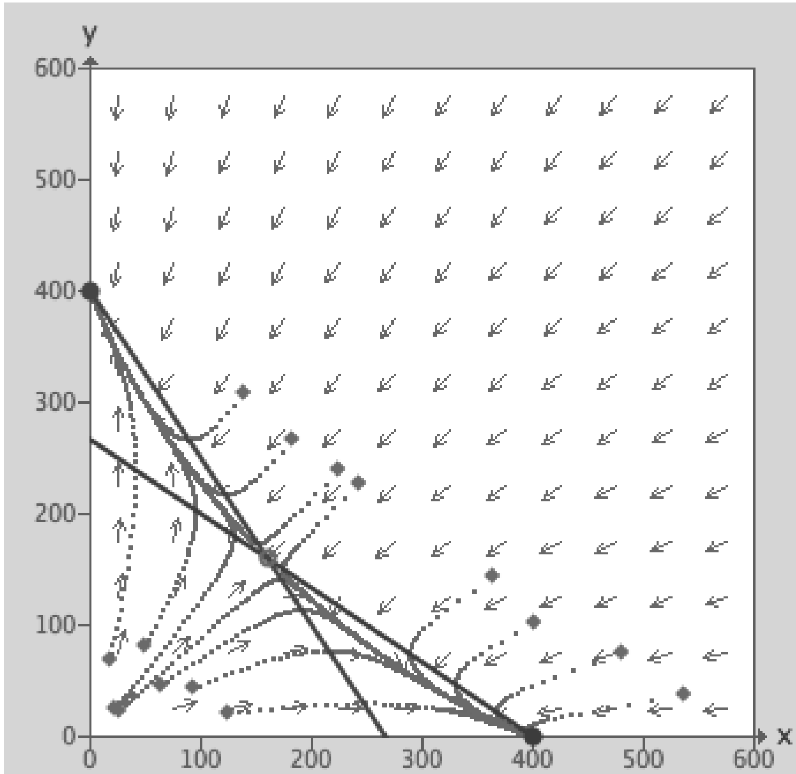
$$\begin{pmatrix} 1 - x/200 - y/200 & -x/200 \\ -y/200 & 1 - x/200 - y/200 \end{pmatrix},$$

and at (400/3, 400/3) this matrix is

$$\begin{pmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{pmatrix}.$$

The determinant of this matrix is $-1/3$. Since the determinant is negative, we know that the point in the trace-determinant plane that corresponds to this matrix lies below the trace axis. So we are indeed in the region where we have a saddle. The phase plane for a, $b > 1$ is shown in Figure 17.3.

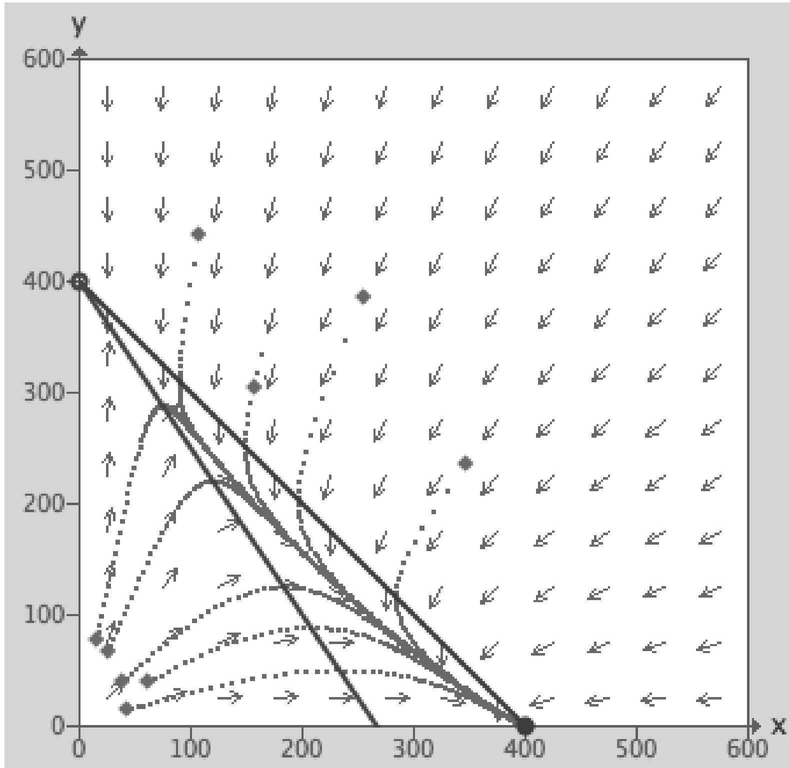
Figure 17.3



Any solution that starts at a point (x_0, y_0) with x_0 and y_0 positive (except for the special solutions that tend to the saddle) tends to either (400, 0) or (0, 400). That is, since both species are very competitive, one will almost surely prevail and the other will go extinct. Which species prevails depends on the initial populations of the 2 species.

Now suppose that species y is much less competitive, so the constant a decreases. Recall that one of the x-nullclines is given by $y = 400/a - x/a$. So the y-intercept of this line is $400/a$. Note that this y-intercept agrees with the y-intercept of the y-nullcline given by $y = 400 - bx$, which is 400 when a is equal to 1. So we see a bifurcation at $a = 1$ since the x- and y-nullclines that do not lie on the axes now meet at the point (0, 400). That is, the coexistence equilibrium point merges with the equilibrium point at (0, 400).

Figure 17.4



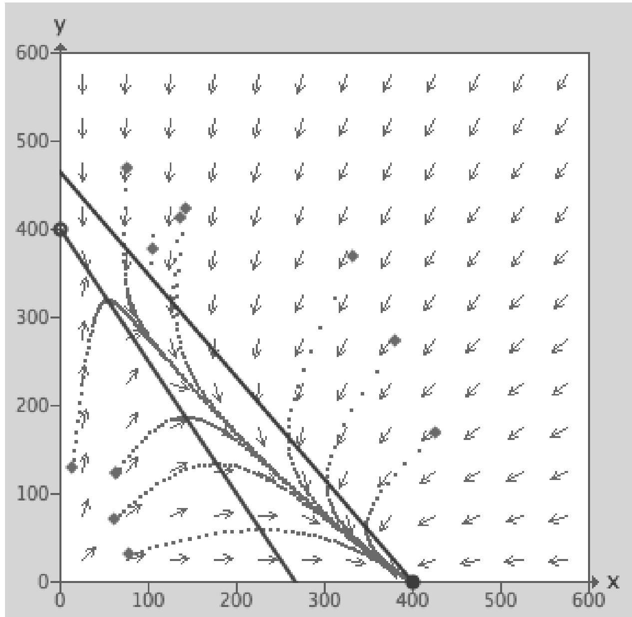
One can check that the Jacobian matrix at the equilibrium point (0, 400) is given by

$$\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}.$$

This matrix has eigenvalues −1 and 0, so linearization does not give us an accurate picture of the phase plane. However, the configuration of the nullclines shows that all solutions that do not lie on the y-axis now tend to the equilibrium point on the x-axis at (400, 0). That is, the species y now goes extinct no matter what the initial point (x_0, y_0) is (as long as x_0 is positive).

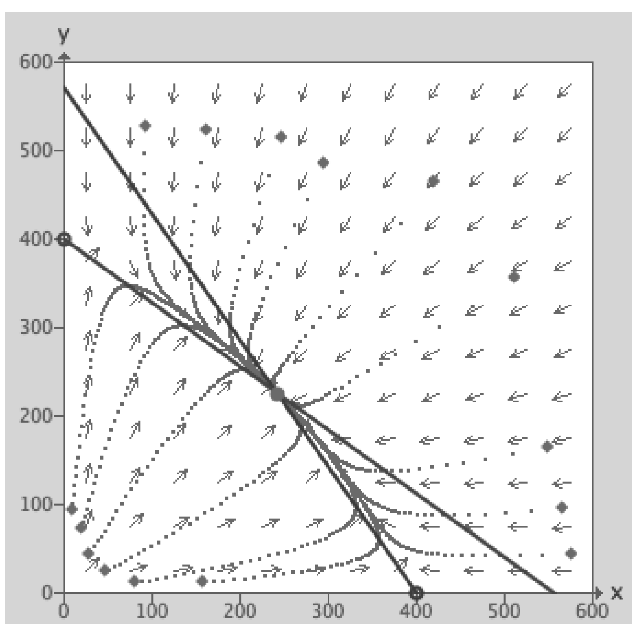
When $a < 1$, the nullclines again indicate that all solutions tend to (400, 0)—the weaker species y remains extinct. Similarly, if b becomes less than 1 while a stays larger than 1, we see that the x-population goes extinct. Below is the phase plane for $a < 1$, $b > 1$.

Figure 17.5



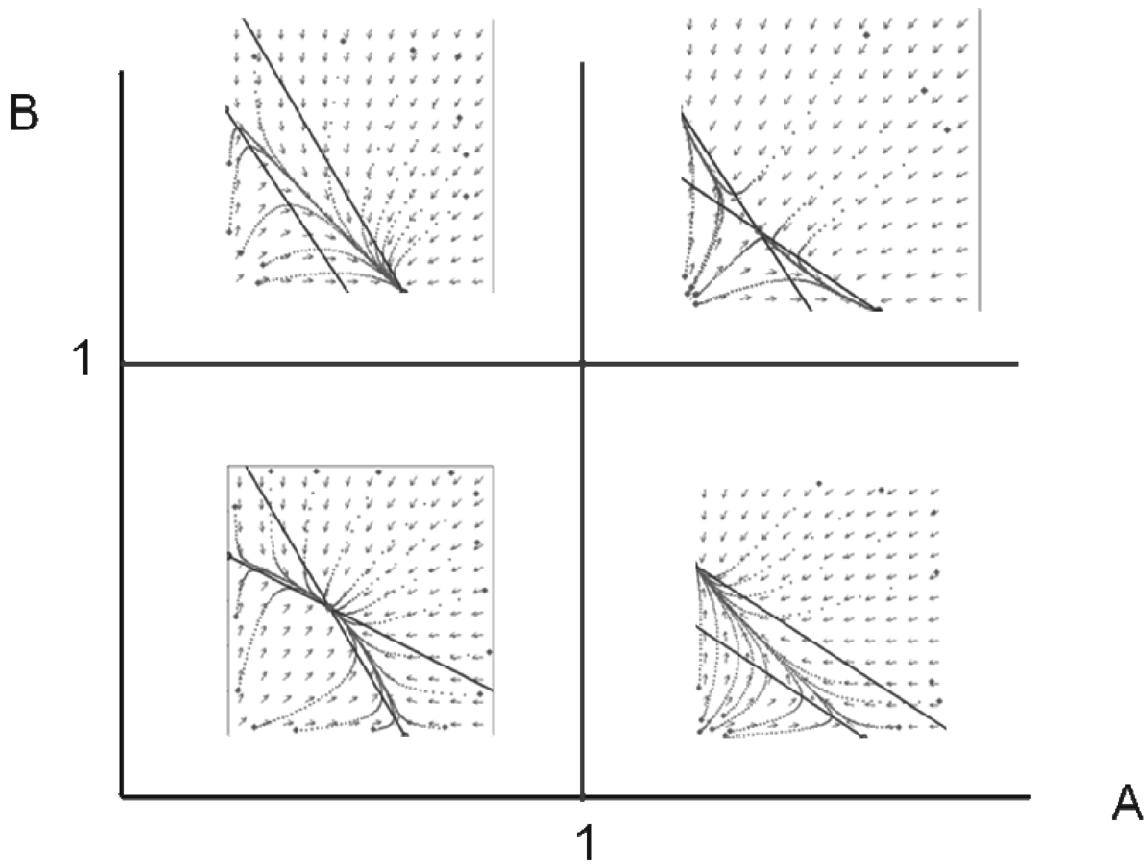
Next, let's look at what happens if both a and b are small (a, $b < 1$). Both of our 2 species are not that competitive. We see another bifurcation as b moves below 1 while a stays at that level. A new coexistence equilibrium point is born. The nullclines, however, indicate that this equilibrium is now a real sink. Indeed, the lack of strong competition leads to coexistence no matter where the 2 populations start out. The below is the phase plane when a, $b < 1$.

Figure 17.6



Finally, we can record all of this information in the analogue of the trace-determinant plane or the bifurcation diagram by plotting the different regions in the *a*-*b* plane where we have different behaviors. This also allows us to see where bifurcations occur.

Figure 17.7



Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 5.1–5.2.

Edelstein-Keshet, *Mathematical Models in Biology*, chap. 6.3.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 11.3.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 6.4.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Competing Species, HPG System Solver.

Problems

1. **a.** Consider a simpler version of the competing species model where we do not include overcrowding:

$$x' = x - xy$$

$$y' = y - xy.$$

Here both x and y are non-negative. First, find all equilibrium points of this system.

b. Compute the Jacobian matrix at each equilibrium point.

c. Determine the types of these equilibria.

d. Determine the nullclines and sketch the regions where the vector field points in different directions.

e. Sketch the phase plane for this system with $x, y \geq 0$.

2. **a.** Verify that the coexistence equilibrium point in the competing species model is indeed a real sink when $a, b = 1/2$.

b. Use linearization to check the types of the equilibrium points that lie on the x- and y-axes for all a- and b-values.

3. **a.** A model due to Beddington and May for the competitive behavior of whales and krill (a small type of shrimp) is

$$x' = x(1 - x) - xy$$
$$y' = y(1 - y/x).$$

Here y is the population of whales. The carrying capacity for the whale population is not constant; rather, it is given by x and therefore depends on the krill population. Find the equilibrium points for this system.

b. Determine the types of the equilibria for the system above.

c. Use nullclines and the computer to paint the picture of the phase plane for this system.

Exploration

Here is a modified competing species model where we now allow harvesting or immigration (the parameter h).

$$x' = x(1 - ax - y)$$
$$y' = y(b - x - y) + h$$

First assume that $h = 0$ (no harvesting). Give a complete synopsis of the behavior of this system by plotting the regions in the a-b plane where different behaviors occur. Then repeat this for the parameters $h > 0$ and $h < 0$. Try to envision the complete picture by viewing the regions in the a-b-h space where different outcomes occur.

Limit Cycles and Oscillations in Chemistry

Lecture 18

For most of the systems of differential equations we have seen thus far, the most important solutions have usually been the equilibrium solutions. But there is another type of solution that often plays an important role in applications: **limit cycles**. These are solutions $Y(t)$ that satisfy $Y(t + A) = Y(t)$ for all values of t and have the additional property that they are isolated. By isolated, we mean that there are no nearby limit cycles; all nearby solutions either spiral in toward the limit cycle or spiral away from it.

As an example of a limit cycle, consider the system

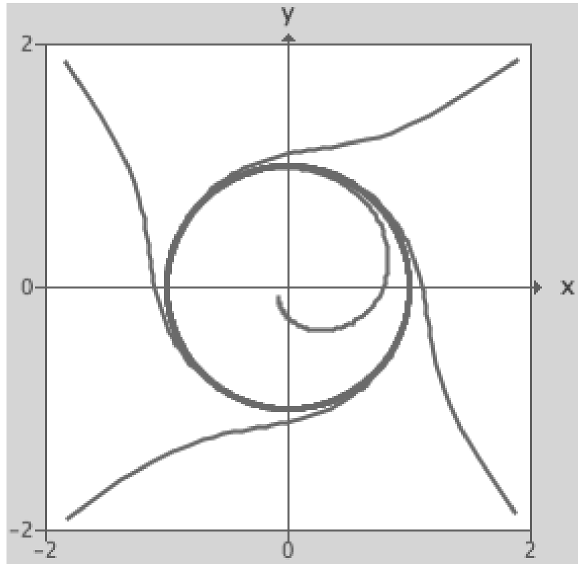
$$x' = -y + x(1 - (x^2 + y^2))$$

$$y' = x + y(1 - (x^2 + y)).$$

Note that this vector field can be broken into a sum of 2 pieces: the linear system $x' = -y$, $y' = x$; and the nonlinear system given by $x' = x(1 - (x^2 + y^2))$, $y' = y(1 - (x^2 + y^2))$. The linear system is a center with solutions traveling around circles centered at the origin. The nonlinear system is a vector field that points directly away from the origin inside the unit circle, points directly toward the origin outside the unit circle, and vanishes on the unit circle. So when we add these 2 vector fields, we find that there is one solution that travels periodically around the unit circle, while all other solutions (except for the equilibrium point at the origin) spiral toward this periodic solution.

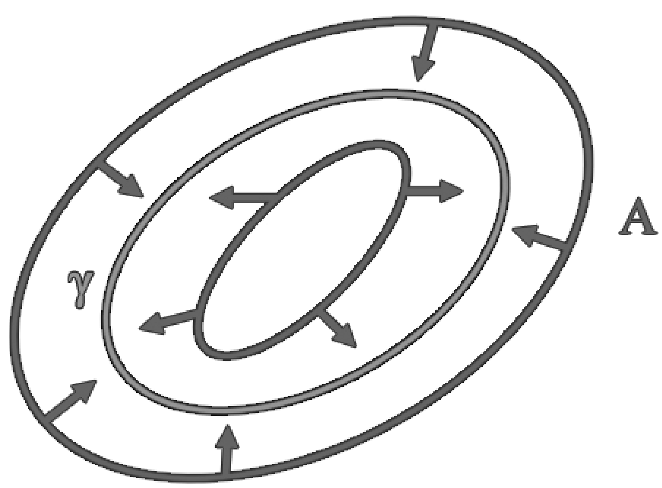
So the solution lying on the unit circle is a stable limit cycle, since nearby solutions move toward this solution. If we change the $+x$ and $+y$ to $-x$ and $-y$ in the second vector field, the solutions now spiral away, so we have an unstable limit cycle.

Figure 18.1



One of the principal tools for finding limit cycles is the Poincare-Bendixson theorem, which says the following: Suppose you have a ring-shaped region *A* in the plane where the vector field points into the interior of *A* along each of the 2 boundaries. Then provided there are no equilibrium points in *A* (and the vector field is sufficiently differentiable), there must be at least one limit cycle in *A*. In the figure below, we see a ring-shaped region *A* with limit cycle γ contained inside.

Figure 18.2



One of the most interesting discoveries in chemistry occurred in the 1950s, when the Russian biochemist Boris Belousov discovered that certain chemical reactions could oscillate back and forth between different states for long periods. Back then, most chemists thought that all chemical reactions settled directly to their equilibrium states. (For the interesting history, see the Winfree article in the Suggested Reading.)

Since that time, many chemical reactions have been found to oscillate. We'll look at a simpler model involving chlorine dioxide and iodide. The nonlinear system is

$$x' = -x + A - \frac{4xy}{1+x^2}$$

$$y' = B\left(x - \frac{xy}{1+x^2}\right),$$

where x and y are the concentrations of iodide and chlorine dioxide, respectively. There are 2 parameters here, A and B. We will simplify things by setting $A = 10$. A little algebra shows that there is a single equilibrium point at (2, 5). Linearizing at this equilibrium point gives the Jacobian matrix

$$\begin{pmatrix} 7/5 & -8/5 \\ 8B/5 & -2B/5 \end{pmatrix}.$$

The trace of the Jacobian matrix is $T = \frac{7}{5} - \frac{2}{5}B$, and the determinant is $D = 2B$. Note that $T = 0$ and $D = 7$ when $B = 3.5$. So in the trace-determinant plane, we see that the linearized equation has a center when $B = 3.5$. When B is a little larger than 3.5, we have $T < 0$, so the equilibrium point is a spiral sink; when B is slightly smaller than 3.5, the equilibrium point is a spiral source. So we have some sort of bifurcation going on when $B = 3.5$. What happens both mathematically and chemically?

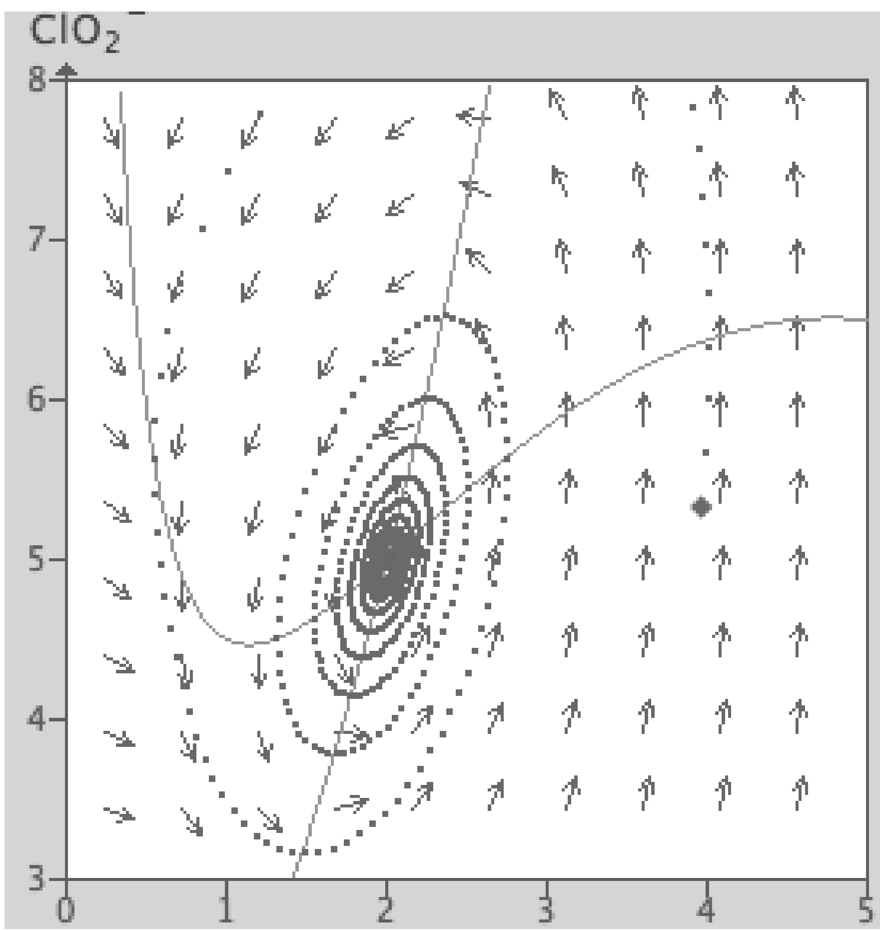
For the whole picture, we plot the nullclines. The x-nullclines are given by $x = 0$ and $y = 1 + x^2$, and the y-nullcline is given by

$$y = \frac{(10 - x)(1 + x^2)}{4x}.$$

Note that the y-nullcline has a vertical asymptote along the y-axis and crosses the x-axis only at $x = 10$. As usual, we compute the vector field in the regions between the nullclines to find what the phase plane looks like.

It is clear in this graph that solutions spiral around the equilibrium point. But how do they do this? When $B > 3.5$, we have a spiral sink at (2, 5), and the computer shows that all solutions spiral into the equilibrium point.

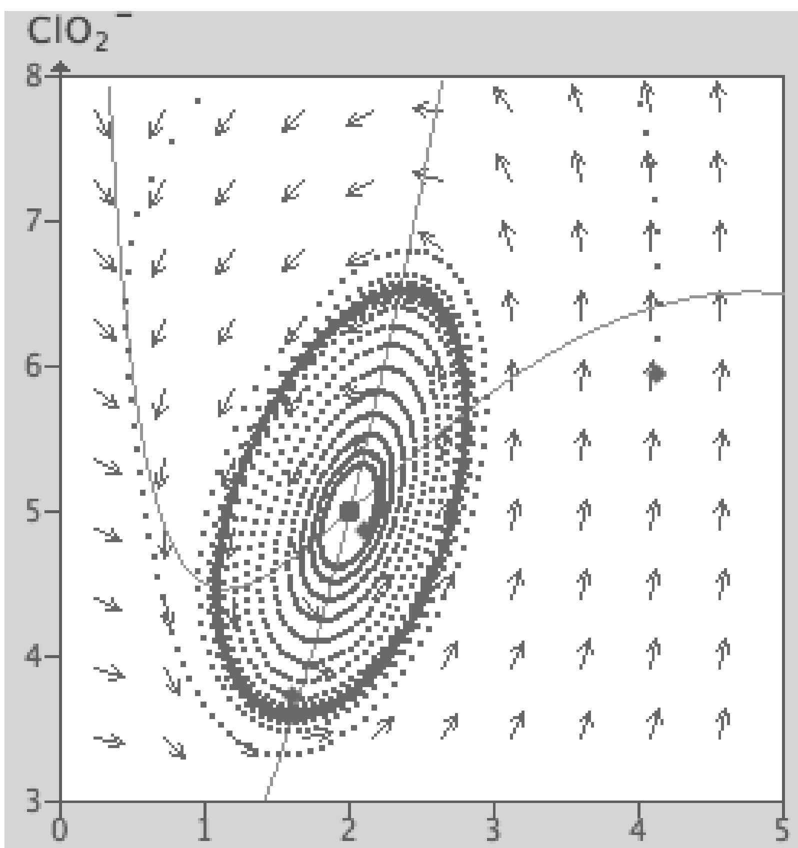
Figure 18.3



When the parameter B goes below 3.5, the equilibrium point changes to a spiral source—but something else happens. If we draw the rectangle bounded by the axes and the lines $x = 10$ and $y = 101$, we see that the vector field points into the rectangle R along all of the boundary points. Meanwhile we have the spiral source at (2, 5), so we can find a small disk D around this point so that the vector field points outside of D at all points on the boundary of D. So we have a ring-shaped region $R - D$ on whose boundary the vector field points into the region $R - D$. By the Poincare-Bendixson theorem, there must be a limit cycle inside $R - D$.

In fact, what has happened is that this system has under gone a **Hopf bifurcation** at $B = 3.5$. Mathematically, when $B > 3.5$, all solutions tend to the spiral sink. But as B passes through 3.5, not only does the spiral sink become a spiral source, but we also see the birth of a limit cycle. In fact, all solutions (except the equilibrium point) now tend to this limit cycle, so this is a stable limit cycle.

Figure 18.4



Chemically, when $B > 3.5$, the reaction oscillates down to its equilibrium state. But when $B > 3.5$, the reaction oscillates back and forth forever.

Important Terms

Hopf bifurcation: A kind of bifurcation for which an equilibrium changes from a sink to a source (or vice versa) and, meanwhile, a periodic solution is born.

limit cycle: A periodic solution of a nonlinear system of differential equations for which no nearby solutions are also periodic. Compared with linear systems, where a system that has one periodic solution will always have infinitely many additional periodic solutions, the limit cycle of a nonlinear system is isolated.

Suggested Reading

Edelstein-Keshet, *Mathematical Models in Biology*, chap. 8.8.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 10.7.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 8.3.

Winfree, “The Prehistory of the Belousov-Zhabotinsky Reaction.”

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Chemical Oscillator, HPG System Solver.

Problems

1. **a.** Does the system

$$x' = x^2 + y^2$$
$$y' = x^2 + y^2$$

have any limit cycles? Explain why or why not.

b. What is the behavior of solutions of this system?

2. Show that the system

$$x' = x - y - x(x^2 + y^2)$$

$$y' = x + y - y(x^2 + y^2)$$

has a periodic solution on the unit circle.

3. Does the system

$$x' = x - x(x^2 + y^2)$$
$$y' = x - y(x^2 + y^2)$$

have a periodic solution on the unit circle?

4. Check that the equilibrium point for the oscillating chemical reaction in the lecture summary is indeed (2, 5).

5. Describe the behavior of solutions of each of the following systems given in polar coordinates.

   a. $r' = r - r^2$, $\theta' = 1$

   b. $r' = r^3 - 3r^2 + 2r$, $\theta' = 1$

   c. $r' = \sin(r)$, $\theta' = -1$

6. Describe the bifurcations that occur in the system given by $r' = ar - r^2 + r^3$, $\theta' = 1$, where we assume that $a > 0$.

Exploration

A chemical reaction due to Schnakenberg is determined by the system

$$x' = x^2y - x - 1/10$$
$$y' = -x^2y + a,$$

where a is a parameter. Find the equilibrium points for this system, and compute the linearization. Do you observe any bifurcations? Can you show the existence of limit cycles for certain parameters?

All Sorts of Nonlinear Pendulums

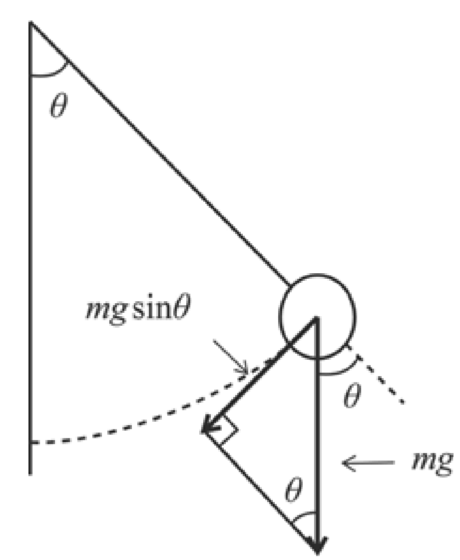
Lecture 19

In this lecture, we introduce several new types of methods used to understand nonlinear systems of differential equations. These include Hamiltonian and Lyapunov functions. In each case, we will concentrate on various nonlinear pendulum equations.

The pendulum equations arise as follows. Suppose we have a light rod of length L with a ball (the bob) of mass m at one end. The other end of the rod is attached to a wall and is free to swing around in a circular motion. Our goal is to describe the motion of the ball. Suppose the position of the center of mass of the ball at time t is given by $\theta(t)$. Let's say that $\theta = 0$ is the downward-pointing direction and that θ increases in the counterclockwise direction.

We will assume that there are 2 forces acting on the pendulum. The first is the constant force of gravity given by mg (where $g \approx 9.8$ m/s^2). Only the force tangent to the circle of motion affects the motion, so this force is $-mg\sin(\theta)$. We determine this with trigonometry.

Figure 19.1



The position of the bob at time t is $(L\sin(\theta(t)), -L\cos(\theta(t)))$. The speed of the bob is then the length of the velocity vector, which is given by $(L\cos(\theta(t))\theta'(t), L\sin(\theta(t))\theta'(t))$. So the length of this vector is $L\theta'$. Similarly, the component of the acceleration vector in the direction of the motion is $L\theta''$. Our second force is then the force due to friction, which we assume as in the case of the mass-spring system is proportional to velocity. So this force is given by $-bL\theta'(t)$, where b is again called the damping constant.

Newton's second law, $F = ma$, then gives the second-order equation for the pendulum:

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{L}\sin(\theta) = 0 .$$

As a system, we get

$$\begin{aligned} \theta' &= v \\ v' &= -\frac{b}{m}v - \frac{g}{L}\sin(\theta). \end{aligned}$$

Note that because of the sine term in the second equation, this is a nonlinear system of differential equations.

Let's first consider the case where there is no friction. This is called the ideal pendulum. We assume as usual that $L = m = 1$, so the system of equations becomes

$$\begin{aligned} \theta' &= v \\ v' &= -g\sin(\theta). \end{aligned}$$

As always, we first find the equilibrium points. Clearly, they are given by $v = 0$ and $\sin(\theta) = 0$. So our equilibria lie at $v = 0$ and $\theta = n\pi$ (where n is any integer). When n is even, this equilibrium point corresponds to the pendulum hanging at rest in the downward position. When n is odd, the pendulum is at rest in its perfectly balanced upward position. There is a conserved quantity for this system of equations. Consider what physicists call the energy function:

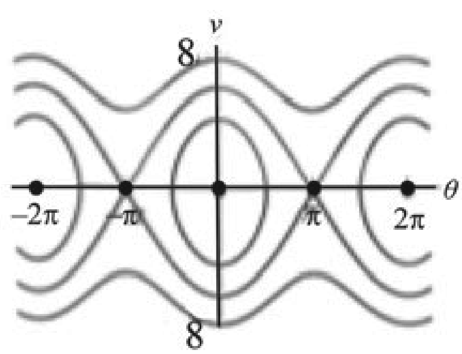
$$H(\theta, v) = \tfrac{1}{2}v^2 - g\cos(\theta) .$$

In our solutions, both θ and v are functions of t, so we can consider the energy function also to be a function of t. And then we can compute its derivative with respect to t. We find that

$$\frac{dH}{dt} = v\,v' - g\sin(\theta)\,\theta' = vg\sin(\theta) - vg\sin(\theta) = 0 .$$

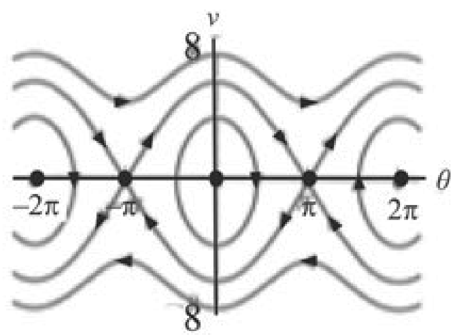
Therefore H must be constant along any of the solution curves of the system. So we just plot the level curves of the function H (i.e., the curves given by H = constant). Then the solutions must lie along these level curves.

Figure 19.2



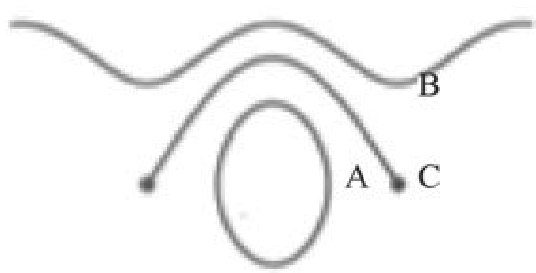
We know that our vector field is everywhere tangent to these level curves, so our solutions must therefore look like the below.

Figure 19.3



Besides our equilibrium solutions where the pendulum is either perfectly balanced in the upright position or hanging motionless in the downward position, we see 3 kinds of other solution curves labeled *A*, *B*, and *C*.

Figure 19.4



Close to the origin (or to any equilibrium point of the form $(2n\pi, 0)$), we see circular motions labeled *A*. These solutions correspond to the pendulum swinging back and forth periodically without ever crossing the upward position θ = π. The second type of solution (labeled *B*) is where the pendulum moves continuously in the counterclockwise direction (or in the clockwise direction if $v < 0$).

The solutions labeled C tend to the upright equilibrium points as t tends to both $\pm\infty$. These upright equilibria are given by $(\pm\pi, 0)$. Using linearization, we can easily check that all of these points are saddle points. So this solution is what is called a **separatrix** (or saddle connection): It tends from one saddle point to another. These are the solutions that you would never see in practice. As time goes on, the pendulum tends to the upright equilibrium without ever passing through $\theta = \pi$.

The ideal pendulum is an example of a Hamiltonian system, named for the Irish mathematician William Rowan Hamilton (1805−1865). A Hamiltonian system in the plane is a system of differential equations determined by a function $H(x, y)$, which is called the Hamiltonian. The system is then given by

$$x' = \frac{\partial H}{\partial y}(x, y)$$
$$y' = -\frac{\partial H}{\partial x}(x, y).$$

H is a conserved quantity since, as before, if we differentiate H with respect to t, we find that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \cdot x'(t) + \frac{\partial H}{\partial y} \cdot y'(t) = \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} + \frac{\partial H}{\partial y} \cdot \left(-\frac{\partial H}{\partial x} \right) = 0 .$$

We have seen another Hamiltonian system earlier, namely, the undamped mass-spring system:

$$y' = v$$

$$v' = -ay.$$

Here the Hamiltonian is given by

$$H(y,v) = \tfrac{1}{2}v^2 + \tfrac{a}{2}y^2$$

since

$$y' = v = \frac{\partial H}{\partial v}$$

and

$$v' = -ay = -\frac{\partial H}{\partial y}.$$

All level curves of H here are just ellipses or circles surrounding the origin (i.e., we have a center equilibrium point at the origin).

Now let's return to the original pendulum equation, but this time we will allow friction. Our second-order equation was

$$\theta'' + b\theta' + g\sin(\theta) = 0,$$

assuming $L = m = 1$ or, as a system,

$$\begin{aligned} \theta' &= v \\ v' &= -bv - g\sin(\theta). \end{aligned}$$

This system is no longer Hamiltonian. But look at our former energy function from the ideal case

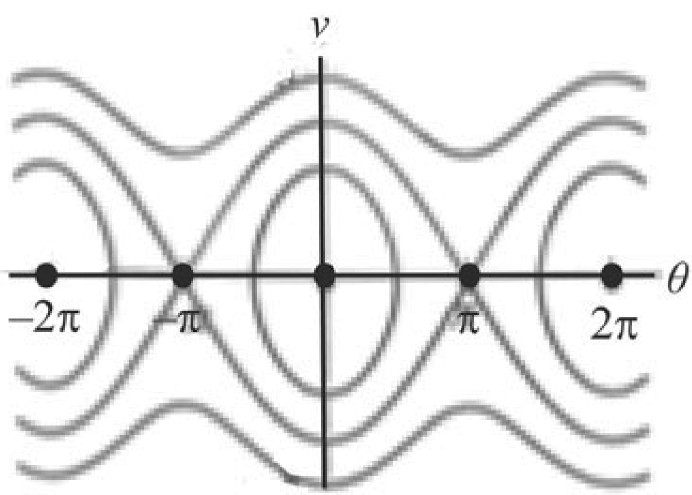
$$H(\theta,v)=\tfrac{1}{2}v^2-g\cos(\theta).$$

We compute

$$\frac{dH}{dt}=v\cdot v'-g\sin(\theta)\,\theta'=v\big(-bv-g\sin(\theta)\big)+g\sin(\theta)\,v=-bv^2\le 0\,.$$

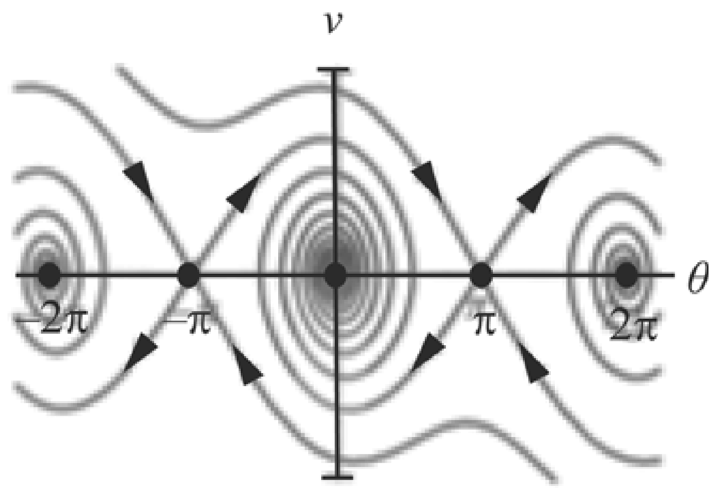
This says that $H(\theta(t),v(t))$ is now a nonincreasing function along solution curves. So our solutions must now descend through the level curves of the function *H*. So we again know what happens to our solutions. Recall that the level curves of *H* looked like the following.

Figure 19.5



Now our solutions must look something like the below.

Figure 19.6



We see that now all solutions tend to the equilibrium point at which the pendulum hangs straight downward. A function like H is called a **Lyapunov function**, which is named for the Russian mathematician Aleksandr Lyapunov (1857–1918).

You may ask, when is a system of the form

$$X' = F(x, y)$$

$$Y' = G(x, y)$$

Hamiltonian? We would need $F(x, y) = \partial H/\partial y$ and $G(x, y) = -\partial H/\partial x$. If such an H exists and is twice continuously differentiable, we would need to see the following.

$$\frac{-\partial G}{\partial y} = \frac{\partial^2 H}{\partial y \partial x} = \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial F}{\partial x}$$

Important Terms

Lyapunov function: A function that is non-increasing along all solutions of a system of differential equations. Therefore, the corresponding solution must move downward through the level sets of the function (i.e., the sets where the function is constant). Such a function can be used to derive the stability properties at an equilibrium without solving the underlying equation.

separatrix: The kind of solution that begins at a saddle point of a planar system of differential equations and tends to another such point as time goes on.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 5.3.

Guckenheimer and Holmes, *Nonlinear Oscillations*, chap. 2.2.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 9.4.

Roberts, *Ordinary Differential Equations*, chap. 10.7.

Strogatz, *Nonlinear Dynamics and Chaos*, chaps. 4.3 and 6.7.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Duffing, Pendulums.

Problems

1. For a typical solution of the ideal pendulum differential equation, at which position is the bob moving fastest?

2. For a typical solution of the ideal pendulum differential equation, at which position is the bob moving slowest?

3. For the ideal pendulum, use linearization to determine the type of the upward equilibrium point.

4. For the ideal pendulum, use linearization to determine the type of the downward equilibrium point.

5. Use linearization to determine the type of the downward equilibrium point for the damped pendulum.

6. Use linearization to determine the type of the upward equilibrium point for the damped pendulum.

7. Is $x' = x^2 + y^2$, $y' = -2xy$ a Hamiltonian system?

8. Consider the linear system $Y' = AY$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Determine conditions on a, b, c, and d that guarantee that this system is Hamiltonian. What types of equilibrium points can occur for such a linear Hamiltonian system?

9. Find the Hamiltonian function for the linear system in problem 8.

Exploration

Duffing's equation is a model for a mass-spring system with a cubic stiffness term. It is also a model for the motion of a magnetoelastic beam mounted between 2 magnets. The second-order equation is given by $y'' + by - y + y^3$. First, assuming the damping constant $b = 0$, show that this is a Hamiltonian system. Second, what happens when b is nonzero? Third, use a computer to investigate the behavior of the Duffing equation when a periodic forcing term, say $F\cos(wt)$, is introduced.

Periodic Forcing and How Chaos Occurs

Lecture 20

In this lecture, we turn to one of the most interesting developments in the modern theory of differential equations: the realization that solutions of systems of differential equations may behave chaotically. Our first example is the periodically forced pendulum. We can force the pendulum in one of 2 ways: moving the pendulum up and down periodically or forcing the bob alternately in the clockwise and counterclockwise directions.

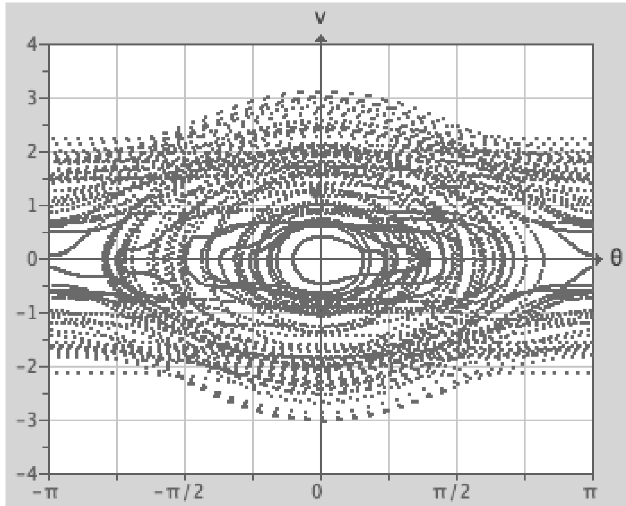
In the first case, the system of differential equations is given by

$$\theta' = v$$
$$v' = -bv - g\sin(\theta) - F\cos(\omega t)\sin(\theta).$$

Here F measures how far we move the pendulum up and down, and $2\pi/\omega$ is the period of the forcing.

In the undamped case, we see that the solution curves meander around the phase plane in a very complicated fashion. Sometimes the pendulum moves around in the clockwise direction; at other times, it moves around in the opposite direction. The question is can you predict 5 minutes from now which way the pendulum will be swinging? The answer is very definitely no; this unpredictability is one of the hallmarks of chaotic behavior.

Figure 20.1



Another ingredient of chaos is what is known as sensitive dependence on initial conditions. Basically, this means the following: If we start the pendulum out at 2 different but very nearby initial points (θ_0, v_0) and (θ_1, v_1), then by continuity, the corresponding solutions start out in very similar fashion. But before long, the 2 solutions diverge from one another, and the corresponding motions of the pendulum are vastly different from one another.

In order to see how we understand this chaos, let's consider the Lorenz system of differential equations. This was the first system of differential equations that was shown to behave in a chaotic fashion. Edward Lorenz (1917–2008) began his career as a mathematician but turned his attention to meteorology while in the army during World War II. In an attempt to understand why meteorologists had a hard time predicting the weather, Lorenz suggested a simplified model for weather. Basically, Lorenz suggested trying to predict the weather on a planet that was surrounded by a single fluid particle. This particle is heated from below and cooled from above, and like the entire atmosphere on Earth, the particle moves around in convection rolls.

The Lorenz system of differential equations that describes this motion is given by

$$x' = a(y - x)$$

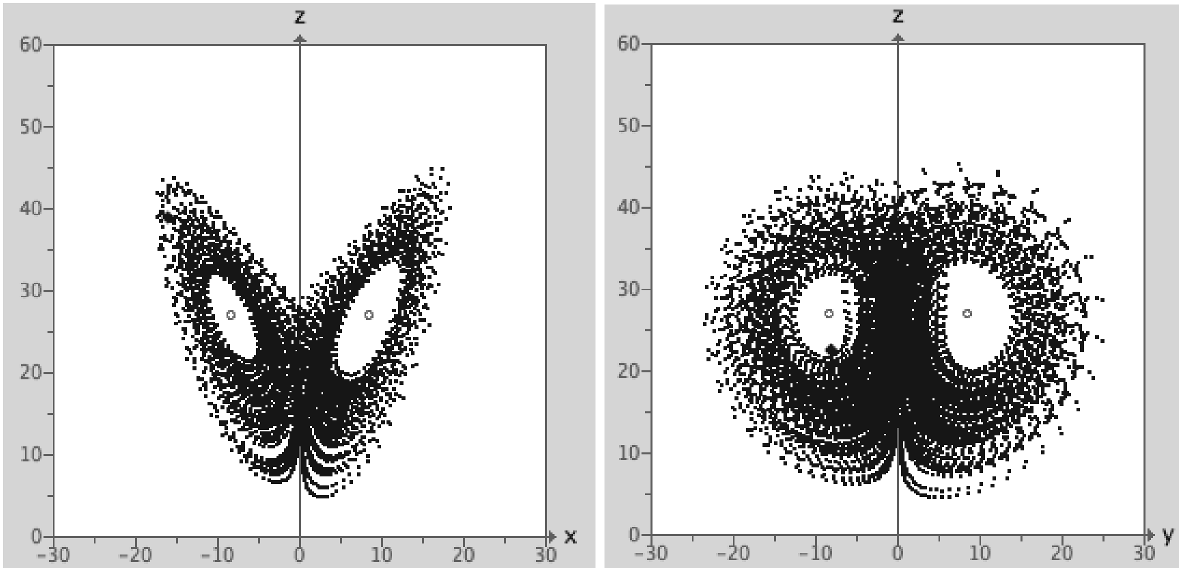
$$y' = Rx - y - xz$$

$$z' = -bz + xy.$$

Here a, b, and r are parameters. For simplicity, let us set $a = 10$ and $b = 8/3$. The remaining parameter $R > 0$ is called the Rayleigh number. We can easily check that there are 3 equilibrium points for this system. One is at the origin, and the other 2 are given by $(x_*, x_*, R-1)$ and $(-x_*, -x_*, R-1)$, where $x_* = \sqrt{8/3\,(R-1)}$. When R is relatively small, most solutions tend to one of the 2 nonzero equilibrium points that correspond to the fluid particle moving periodically in either the clockwise or counterclockwise direction.

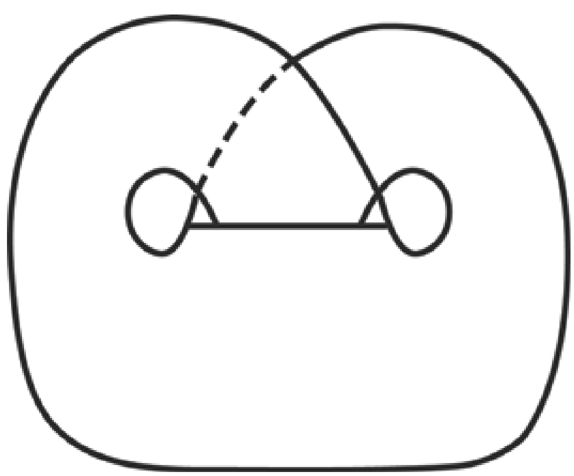
But as R increases, suddenly the solutions no longer tend to the equilibria. Rather, they tend to circle around these 2 points in a kind of haphazard fashion. If we choose 2 nearby initial conditions, just as in the forced pendulum case, the corresponding solutions very quickly diverge from one another, and we again have sensitive dependence on initial conditions. The figures below are 2 views of the same orbit: one projected into the x-z plane (left), the other into the y-z plane (right). This type of sensitive dependence has been called the butterfly effect.

Figure 20.2



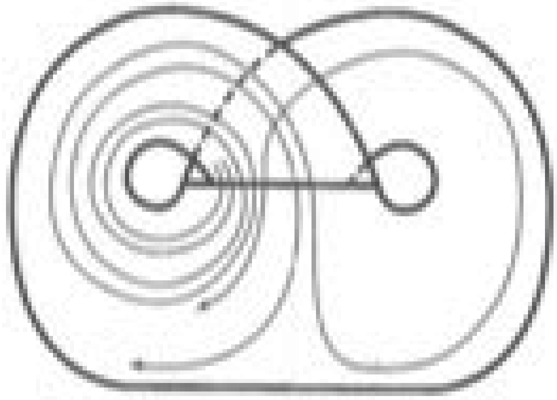
Since our solutions lie in 3-dimensional space, it is difficult to see what is actually happening. It appears that the solutions are tending toward a 2-dimensional object called the Lorenz template (or the Lorenz mask). Note that 2 lobes of the template are joined along a straight line, one lobe in front and the other in the rear.

Figure 20.3



Then it appears that solutions wind around this template, looping around the 2 holes and crossing the central line over and over again.

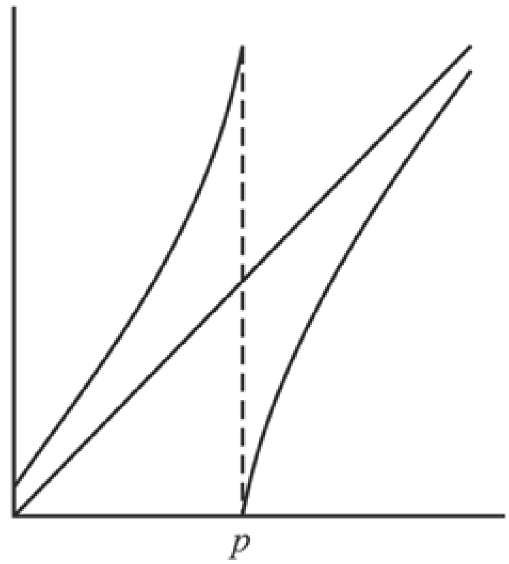
Figure 20.4



Technically, this cannot happen because we would then have different solutions that merge into the same solution, contradicting the existence and uniqueness theorem (which we discussed earlier). But the fact is that solutions do come very close to an object that is similar in shape to this template—only there are infinitely many leaves that are bundled closely together. This object is known as the Lorenz attractor.

Most solutions on this template return over and over again to the central line as depicted above. So instead of looking for the entire intricate solution of the Lorenz system, what we can do is look at how this solution returns over and over again to this line denoted by L. So we can think of these solutions as being determined by the first return function defined on L. This function assigns to each point p on L the next point along the solution through p that lies on L. So this gives us the first return function F: $L \to L$. The approximate shape of the graph of F is below.

Figure 20.5



Now the solution starting at p can be tracked by iterating this function. That is, given p, we first compute $F(p)$ to find the next return to L. Then we compute $F(F(p))$ to get the second return, and so forth. We denote this second iterate of F by $F^2(p)$. Then $F^3(p)$ gives the third return, and so forth. The collection of points $F^n(p)$ is called the **orbit** of p under F, and determining how this orbit behaves gives us a good idea of what is happening to the orbits on the Lorenz attractor.

Amazingly, many simple functions have chaotic behavior when iterated. For example, look at what happens to the orbits of 0 and 0.0001 for the quadratic function $x^2 - 2$ or the orbits of .5 and .5001 for another quadratic, $4x(1 - x)$.

Important Term

orbit: In the setting of a difference equation, an orbit is the sequence of points x_0, x_1, x_2, ... that arise by iteration of a function F starting at the seed value x_0.

Suggested Reading

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 2.5 and 8.5.

Guckenheimer and Holmes, *Nonlinear Oscillations*, chap. 2.3.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 14.

Strogatz, *Nonlinear Dynamics and Chaos*, chap. 9.

Relevant Software

Blanchard, Devaney, and Hall, *DE Tools*, Butterfly Effect, Lorenz Equations, Pendulum Sensitive Dependence, Pendulums.

Problems

1. **a.** Consider the Duffing oscillator equation with no forcing

$$y' = v$$

$$v' = y - y^3.$$

What are the equilibrium points for this system?

b. Show that this system is Hamiltonian with Hamiltonian function

$$H(y, v) = v^2/2 - y^2/2 + y^4/4.$$

c. Sketch the level curves of this Hamiltonian function.

d. Describe the motion of the slender steel beam in the different cases that arise.

2. Show that for the Lorenz system, there is a straight line solution converging to the origin along the z-axis.

3. Show that the function

$$L(x, y, z) = x^2 + 10y^2 + 10z^2$$

is a Lyapunov function for the Lorenz system when $R < 1$.

4. Given the results of the problem above, what can you say about solutions when $R < 1$?

5. Let $V(x, y, z) = Rx^2 + 10y^2 + 10(z - 2R)^2$. Note that $V(x, y, z) = c$ defines an ellipsoid centered at $(0, 0, 2R)$ in 3-dimensional space. Show that we may choose c large enough so that V is decreasing along any solution curve that starts outside the ellipsoid given by $V(x, y, z) = c$.

6. What does this say about solutions of the Lorenz system that start far from the origin?

Exploration

Consider the Rossler system given by

$$x' = -y - z$$

$$y' = x + y/4$$

$$z' = 1 + z(x - c)$$

where $0 < c < 7$. First find all equilibrium points for this system. Describe the bifurcation that occurs at $c = 1$. Then use a computer to investigate what else happens for this system as c increases. Are there any similarities to the Lorenz system?

Understanding Chaos with Iterated Functions

Lecture 21

As we discussed in the previous lecture, the way mathematicians have begun to understand the chaotic behavior that occurs in higher-dimensional systems of differential equations is often by iterating a first return map.

Here is another way iteration arises. Recall the limited population growth model (a.k.a. the logistic population model) that we saw earlier. This was the differential equation $y' = ky(1 - y/N)$ that we solved in multiple different ways. Here is a variation on this theme—the discrete logistic population model. Suppose x_n denotes the population of some species in year n (or, for example, generation n). The discrete logistic population model is given by

$$x_{n+1} = kx_n(1 - x_n/N).$$

That is, the population given in year $n + 1$ is just $kx_n(1 - x_n/N)$, where k and N are constants that depend on the particular species. Here, N is the maximal population rather than the ideal population. For simplicity, let's look at the equation

$$x_{n+1} = kx_n(1 - x_n).$$

Here x_n represents the fraction of the maximal population. This is an example of a **difference equation**. To find the population in ensuing years, we simply start with an initial population (or seed) x_0 and plug it into the function $f(x) = kx(1 - x)$. Out comes x_1. Then we do the same with x_1 to generate x_2. We do this over and over to produce $x_0, x_1, x_2, ...$, the so-called orbit of the seed x_0.

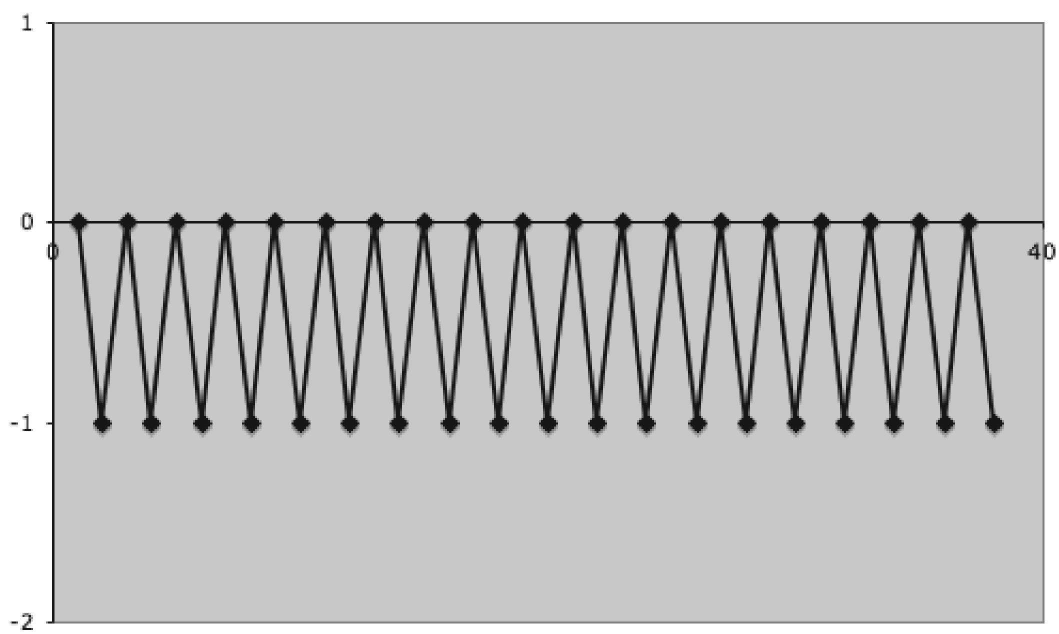
For difference equations, let's consider more generally $x_{n+1} = f(x_n)$ for some function f. We iterate f to produce the orbit of some seed x_0. For example, suppose $f(x) = x^2$. Then the orbit of 0 is 0, 0, 0, ..., so we say that 0 is a **fixed point** for f. We see that -1 is an eventually fixed point since $f(-1) = 1$,

which is also fixed. If $0 < |x| < 1$, then the orbit of x tends to the fixed point at 0. But if $|x| > 1$, repeated squaring sends the orbit of x off to infinity. So we understand the fate of all orbits in this simple case.

As another example, if $f(x) = x^2 - 1$, the points 0 and -1 lie on a cycle of period 2 since $f(-1) = 0$ while $f(0) = -1$. We also say that 0 and -1 are periodic points with prime period 2. More generally, x_0 is a periodic point of prime period n if $x_n = x_0$ and n is the smallest positive integer for which this is true.

As with earlier parts of this course, we can visualize orbits in several different ways. One way to do this is to plot a time series for the orbit. For example, for our cycle of period 2 for $f(x) = x^2 - 1$, the time series representation of the orbit would be the following.

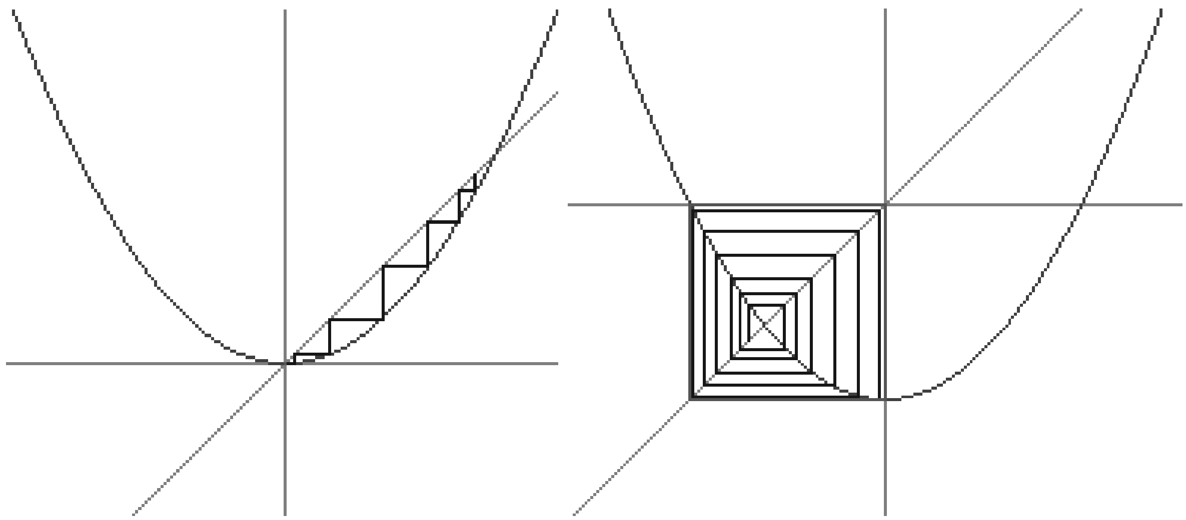
Figure 21.1



A second way to visualize orbits is to use graphical analysis. Given the function f, we plot its graph and then superimpose the line $y = x$ (the diagonal line) on this graph. Given a seed x_0, we first draw a vertical line

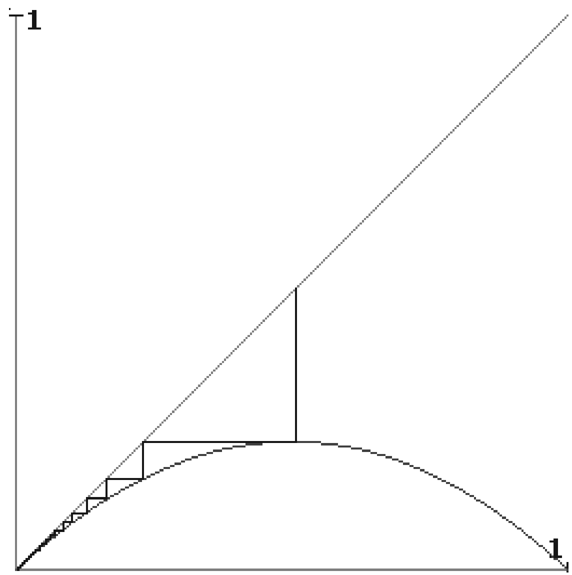
from the point (x_0, x_0) on the diagonal to the point $(x_0, f(x_0)) = (x_0, x_1)$ on the graph. Then we draw a horizontal line back to the diagonal yielding (x_1, x_1). We continue to (x_1, x_2) on the graph and over to (x_2, x_2) on the diagonal. For example, below is the graphical analysis representation of the orbit of 0.9 under $f(x) = x^2$ (left) and the orbit of -0.7 under $f(x) = x^2 - 1$ (right). Note that the orbit of -0.7 in this second graph tends to the 2-cycle at 0 and -1.

Figure 21.2



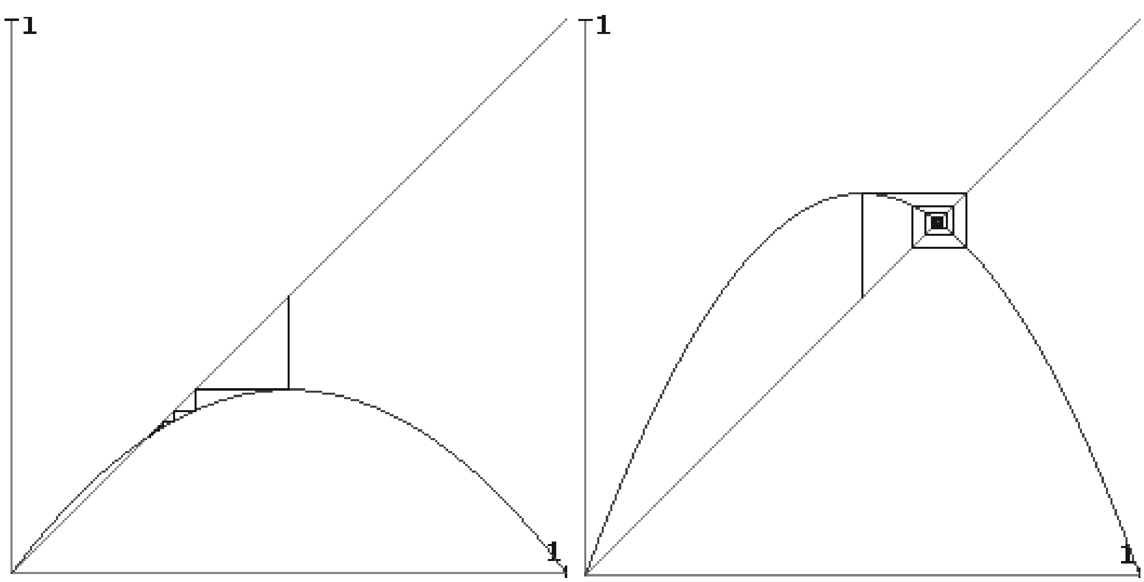
Unlike the logistic differential equation where all solutions essentially do the same thing, for the discrete logistic model, orbits can do many different things. We will use the orbit of the critical point for the function $f(x) = kx(1 - x)$ from now on. A critical point is a point where the derivative of the function vanishes, so when $f(x) = kx(1 - x)$, we have $f'(x) = k - 2kx$, so the only critical point is $x = 1/2$. When $0 < k < 1$, the critical point tends to 0, which is a fixed point.

Figure 21.3



When $1 < k < 3$, graphical analysis shows that the critical orbit now tends to a nonzero fixed point. This point is the nonzero root of the equation $kx(1 - x) = x$, namely $(k - 1)/k$.

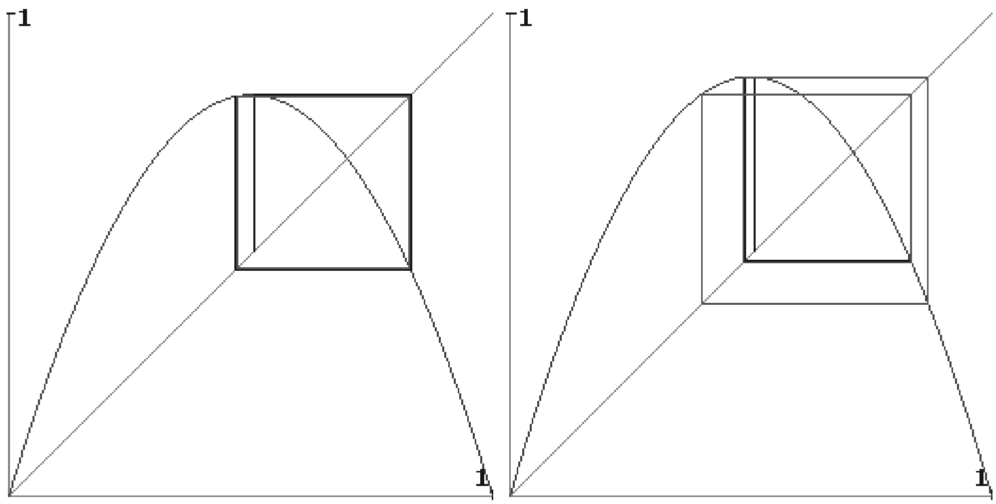
Figure 21.4



$k = 1.3$ $k = 2.75$

For $k = 3.3$, the critical orbit now tends to a 2-cycle, but when $k = 3.47$ the critical orbit tends to a 4-cycle.

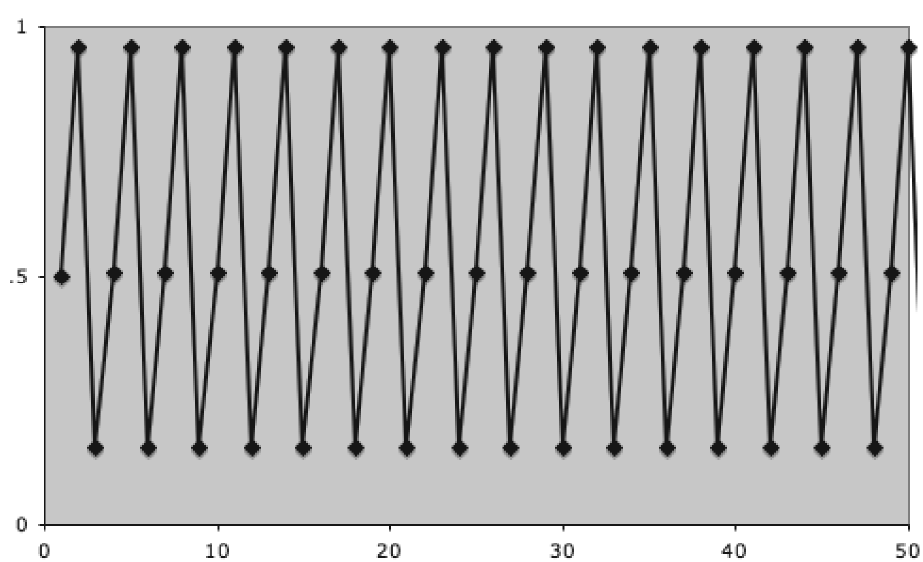
Figure 21.5



$k = 3.3$ $k = 3.47$

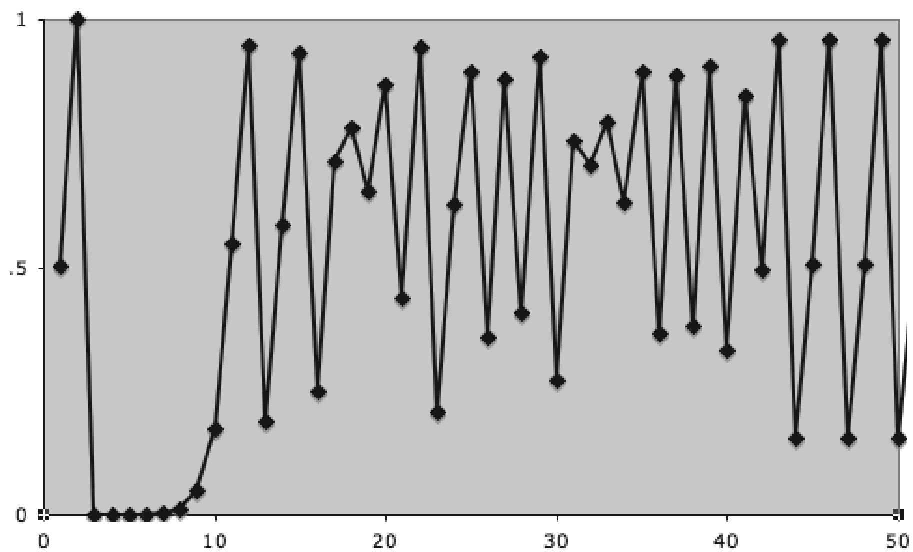
When $k = 3.83$, the critical orbit tends to a 3-cycle. The time series for this critical orbit is below.

Figure 21.6



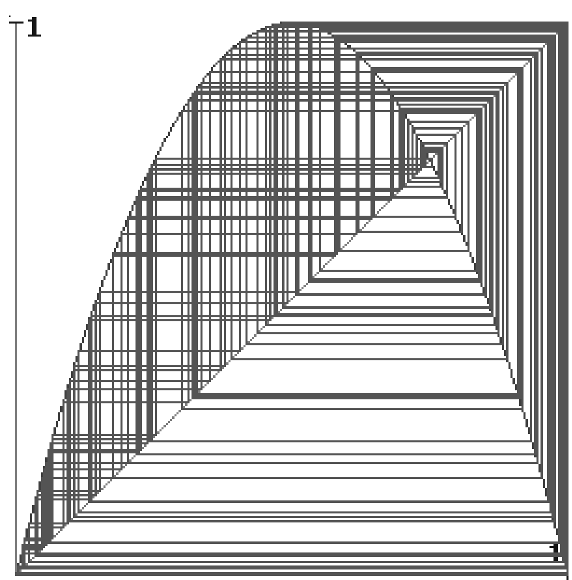
When $k = 4$, things go crazy. The orbit of 1/2 is quite simple: It is 1/2, 1, 0, 0, 0, …, so the orbit is eventually fixed. But any nearby orbit behaves vastly differently. The orbit of 0.5001 as a time series is below.

Figure 21.7



And here it is using graphical analysis.

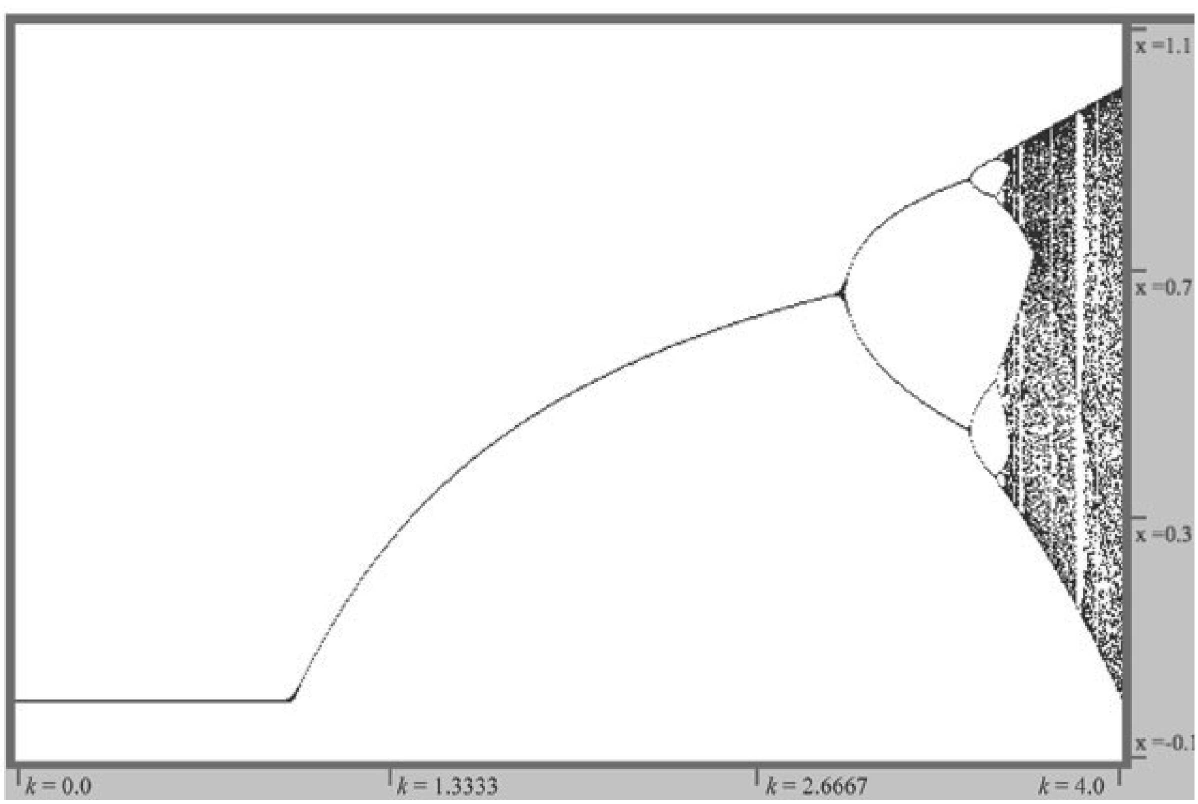
Figure 21.8



We find similar behaviors when we iterate the quadratic function $x^2 + c$. When $c = 0$, the orbit of 0 is fixed; $c = -1$, a 2-cycle; $c = -1.3$, a 4-cycle; $c = -1.38$, an 8-cycle; $c = -1.76$, a 3-cycle. And again we get chaotic behavior when $c = -2$.

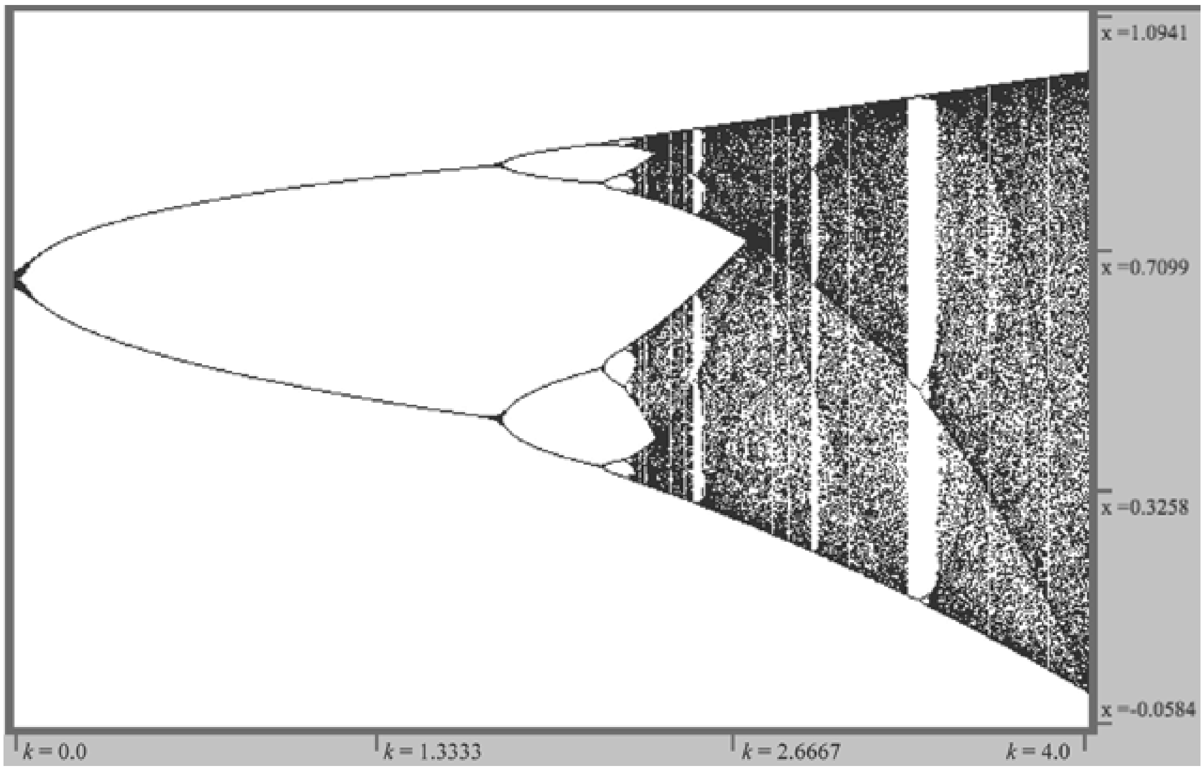
Clearly, lots of things are happening when we change the parameter k in the logistic function or c in $x^2 + c$. To see all of this, we plot the **orbit diagram** for this function. In this graphic, we plot the parameter k horizontally with $0 \le k \le 4$. Along the vertical line above a given k-value, we plot the eventual orbit of the critical point 1/2. By "eventual," we mean that we iterate, say, 200 times but only display the last 100 points on the orbit. That is, we throw away the transient behavior.

Figure 21.9



And here is a magnification of the orbit diagram with k in the interval $3 \le k \le 4$.

Figure 21.10



Important Terms

difference equation: An equation of the form $y_{n+1} = F(y_n)$. That is, given the value y_n, we determine the next value y_{n+1} by simply plugging y_n into the function F. Thus, the successive values y_n are determined by iterating the expression $F(y)$.

fixed point: A value of y_0 for which $F(y_0) = y_0$. Such points are attracting if nearby points have orbits that tend to y_0; repelling if the orbits tend away from y_0; and neutral or indifferent if y_0 is neither attracting or repelling.

orbit diagram: A picture of the fate of orbits of the critical point for each value of a given parameter.

Suggested Reading

Alligood, Sauer, and Yorke, *Chaos*, chap. 1.5.

Blanchard, Devaney, and Hall, *Differential Equations*, chap. 8.1.

Devaney, *A First Course in Chaotic Dynamical Systems*, chap. 3.

Hirsch, Smale, and Devaney, *Differential Equations*, chaps. 15.1–15.3.

Relevant Software

Nonlinear Web, http://math.bu.edu/DYSYS/applets/nonlinear-web.html

Orbit Diagram for $Cx(1-x)$, http://math.bu.edu/DYSYS/applets/bif-dgm/Logistic.html

Problems

1. **a.** Find all fixed points for the function $F(x) = x^2$.

b. What is the fate of all other orbits for this function?

2. What is the fate of all orbits of $F(x) = x^2 + 1$?

3. **a.** What is the fate of all orbits of $F(x) = ax$ where $0 < a < 1$?

b. What is the fate of orbits of this function for other values of a?

4. Find all fixed points for the logistic family $kx(1 - x)$ with $k > 0$.

5. Find the cycle of period 2 for the function $F(x) = -x^3$. Is there a 2-cycle for the function $F(x) = x^3$?

6. Let $D(x)$ denote the doubling function defined on the interval $0 \leq x \leq 1$ and given by

$2x$ if $0 \leq x < 1/2$ and

$2x - 1$ if $1/2 \leq x \leq 1$.

Show that the seeds 1/3, 1/7, and 1/15 lie on cycles; and compute their periods.

7. Find all points that lie on cycles of prime period 2, 3, and 4 for the doubling function.

8. Draw the graph of $D(x)$, $D^2(x)$, and $D^3(x)$. How many times does the graph of $D^n(x)$ cross the diagonal? Find all points that lie on cycles of period n for the doubling function (not necessarily prime period n).

Exploration

Using the applets below, investigate the orbit diagrams for the functions $x^2 + c$, $cx(1 - x^2/3)$, and $c\sin(x)$. Do you observe any similarities in these diagrams as you zoom in?

Orbit Diagram for $x^2 + c$:

http://math.bu.edu/DYSYS/applets/bif-dgm/Quadratic.html

Bifurcations of the cubic map:

http://math.bu.edu/DYSYS/applets/bif-dgm/Cubic.html

Bifurcations of the sine map:

http://math.bu.edu/DYSYS/applets/bif-dgm/Sine.html

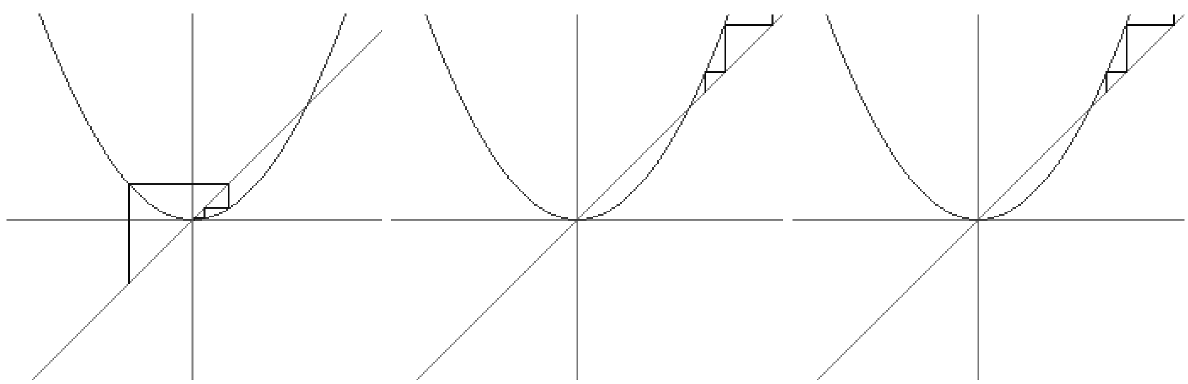
Periods and Ordering of Iterated Functions

Lecture 22

As we saw when we studied first-order differential equations, there are 3 different types of equilibrium points—sinks, sources, and nodes. The same is true for iterated functions: There are 3 different types of fixed points—attracting, repelling, and neutral (or indifferent).

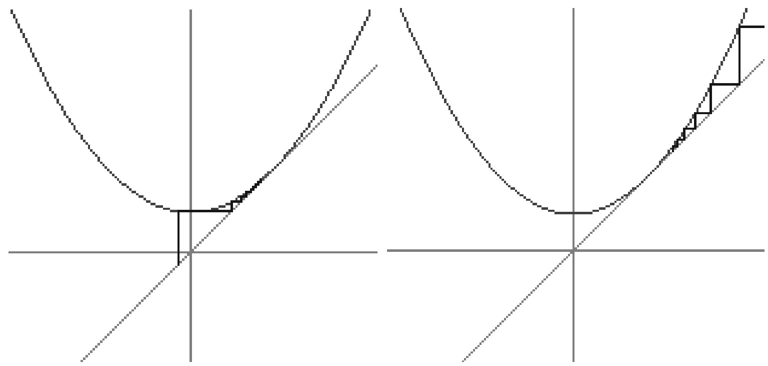
A fixed point is attracting if nearby seeds to the fixed point have orbits that tend to the fixed point. It is repelling if nearby orbits tend away from the fixed point. And it is neutral if neither of the above cases occurs. For example, $f(x) = x^2$ has an attracting fixed point at $x_0 = 0$ and a repelling fixed point at $x_0 = 1$. This is easily seen with graphical analysis.

Figure 22.1



Also, $f(x) = x^2 + 1/4$ has a neutral fixed point at $x_0 = 1/2$ since nearby orbits to the left of the fixed point tends toward it, while orbits to the right of the fixed point tend away from it.

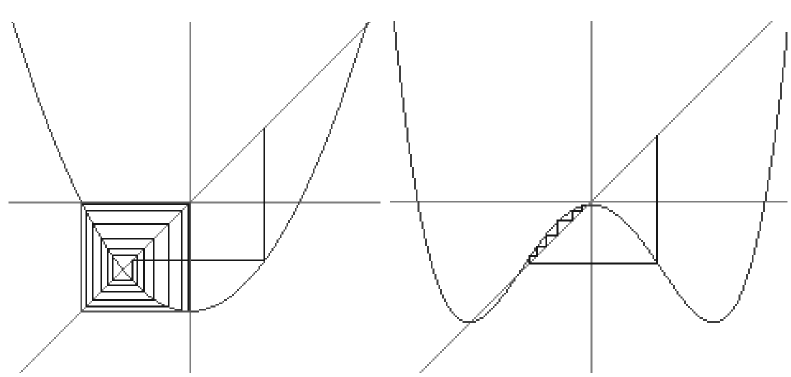
Figure 22.2



We can use calculus to determine the types of fixed points. Suppose we have a fixed point x_0, so $f(x_0) = x_0$. If $|f'(x_0)| < 1$, then graphical analysis shows that this fixed point is attracting. If $|f'(x_0)| > 1$, then x_0 is repelling. If $f'(x_0) = \pm 1$, then we get no information; x_0 could be attracting, repelling, or neutral.

If we have a periodic point x_0 of period n, then we know that $f^n(x_0) = x_0$. So this periodic point is attracting or repelling depending on whether $|(f^n)'(x_0)|$ is less than or greater than 1. For example, for the periodic points 0 and -1 of period 2 for $f(x) = x^2 - 1$, we have $f^2(x) = x^4 - 2x^2$, so $(f^2)'(0) = (f^2)'(-1) = 0$. Therefore these periodic points are attracting. Graphical analysis of both $f(x)$ and $f^2(x)$ shows this as well.

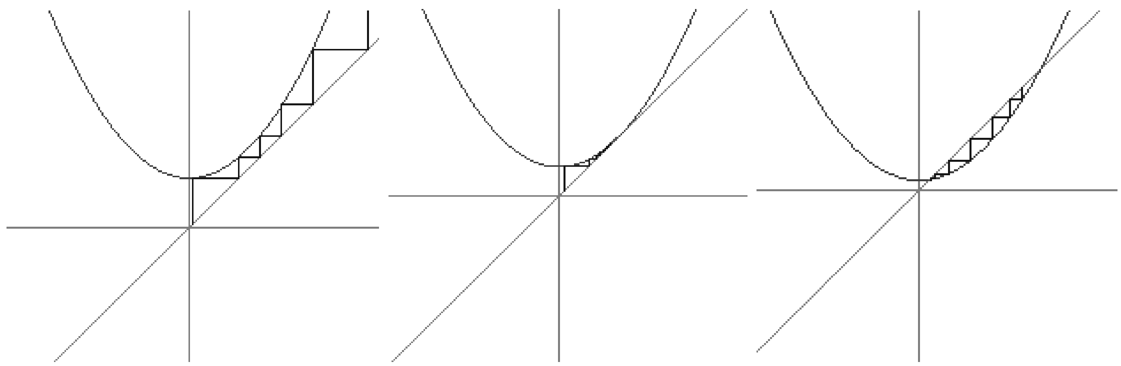
Figure 22.3



$f(x) = x^2 - 1$ $\qquad\qquad$ $f^2(x) = x^4 - 2x^2$

Bifurcations also arise when we iterate functions. For example, consider $f(x) = x^2 + c$. There is a bifurcation when $c = 1/4$ as is seen in the graphs when we pass from $c > 1/4$ to $c < 1/4$. There are no fixed points when $c > 1/4$; a single (neutral) fixed point when $c = 1/4$; and 2 fixed points (1 attracting, 1 repelling) when c is slightly below 1/4. This type of bifurcation is called a saddle-node bifurcation, just as in the case of first-order differential equations.

Figure 22.4



$c > 1/4$ $\qquad$ $c = 1/4$ $\qquad$ $c > 1/4$

There is a very different type of bifurcation that we saw many times in the orbit diagram. This is the period doubling bifurcation. What happens at this bifurcation is an attracting periodic cycle of period n suddenly becomes repelling, and at the same time, an attracting cycle of period $2n$ branches away. For example, consider again $f(x) = x^2 + c$. When $c < 1/4$, we always have 2 fixed points at

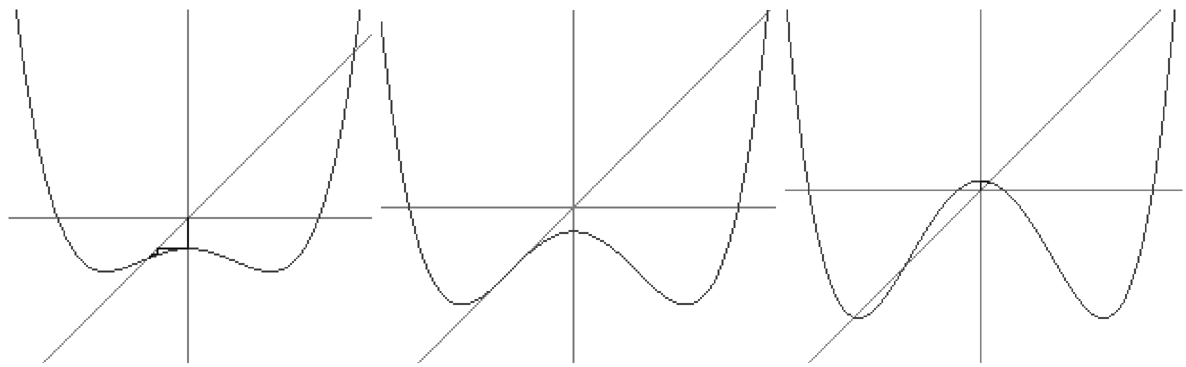
$$x_{\pm} = \frac{1}{2} \pm \frac{\sqrt{1-4c}}{2}.$$

The fixed point at x_+ is always repelling. At x_- we have

$$f'(x_-) = 1 - \sqrt{1-4c}.$$

So this fixed point is attracting if $-3/4 < c < 1/4$ and repelling if $c < -3/4$. We have $f'(x_-) = -1$ if $c = -3/4$. More importantly, graphical analysis shows that a new periodic point of period 2 is born when c goes below $-3/4$. Equivalently, one can check that the equation $f^2(x) = x$ has only 2 real roots when $c > -3/4$ (the 2 fixed points) whereas there are 4 roots when $c < -3/4$. This is the period doubling bifurcation.

Figure 22.5



$c > -3/4$ $\quad$ $c = -3/4$ $\quad$ $c < -3/4$

One of the most amazing theorems dealing with iteration of functions on the real line is a result proved by Alexander Sharkovsky in the 1960s. This theorem says the following. Consider the following ordering of the natural numbers: First list all the odds (except 1) in increasing order, then list 2 times the odds (again except 1) in increasing order, then 4 times the odds, 8 times the odds, and so on. When you have done this, the only missing numbers are the powers of 2, which we then list in decreasing order. Below is the Sharkovsky ordering.

$3 \Rightarrow 5 \Rightarrow 7 \Rightarrow 9 \Rightarrow \ldots$

$2 \bullet 3 \Rightarrow 2 \bullet 5 \Rightarrow 2 \bullet 7 \Rightarrow 2 \bullet 9 \Rightarrow \ldots$

$4 \bullet 3 \Rightarrow 4 \bullet 5 \Rightarrow 4 \bullet 7 \Rightarrow \ldots$

$8 \bullet 3 \Rightarrow 8 \bullet 5 \Rightarrow 8 \bullet 7 \Rightarrow \ldots$

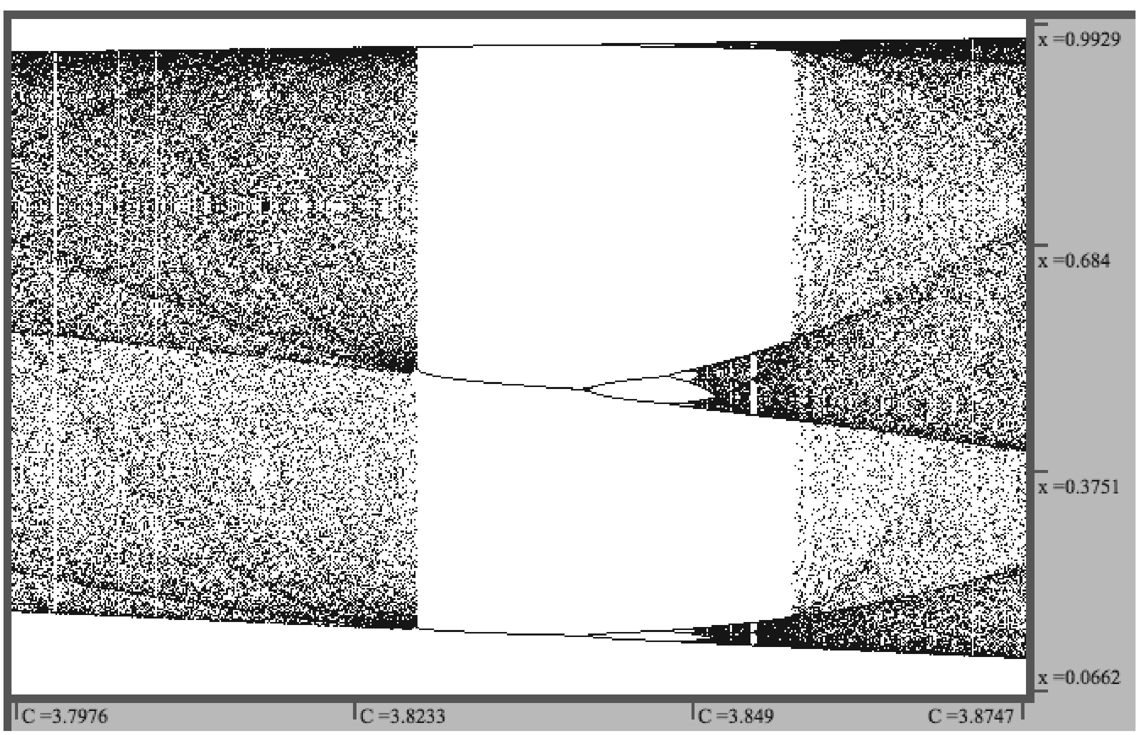
......

$\ldots 2^3 \Rightarrow 2^2 \Rightarrow 2 \Rightarrow 1$

Sharkovsky's theorem says that suppose you have a function that is continuous on the real line. If this function has a periodic point of prime period *n*, then it must also have a periodic point of prime period *k* for any integer that follows *n* in the Sharkovsky ordering. In particular, if a continuous function has a periodic point of period 3, then it must also have a periodic point of all other periods!

Now go back to the orbit diagram for the logistic function. Toward the right, we see a little window that had, at least initially, what appears to be a cycle of period 3. By Sharkovsky, there must be infinitely many other periodic points in this window. Why do we not see them?

Figure 22.6. The period-3 window in the orbit diagram.



Well, the answer is that only the period-3 cycle is attracting; all the other cycles are repelling. The reason for this is another amazing fact about iteration of polynomials (and, in fact, what are called analytic functions). If you have an attracting or neutral periodic cycle, then the orbit of one of the critical points must be attracted to it. This means that for the logistic map (where there is only one critical point, $x = 1/2$), we can have at most one attracting cycle. Everything else must be repelling.

Suggested Reading

Alligood, Sauer, and Yorke, *Chaos*, chaps. 1.3–1.4.

Blanchard, Devaney, and Hall, *Differential Equations*, chaps. 8.2–8.3.

Devaney, *A First Course in Chaotic Dynamical Systems*, chap. 5.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 15.2.

Relevant Software

Nonlinear Web, http://math.bu.edu/DYSYS/applets/nonlinear-web.html

Orbit Diagram for $Cx(1-x)$, http://math.bu.edu/DYSYS/applets/bif-dgm/Logistic.html

Problems

1. Find the fixed points for $F(x) = x^3$ and determine their type.

2. **a.** Is the cycle of period 2 for $F(x) = x^2 - 1$ given by 0 and -1 attracting or repelling?

b. Illustrate this using graphical analysis.

3. **a.** For the function $F(x) = ax$, what is the type of the fixed point at 0? (This type will depend upon a.)

   b. At which a-values does this family of functions undergo a bifurcation?

4. Determine the values of k for which the fixed points in the unit interval for the logistic family $kx(1 - x)$ with $k > 0$ are attracting, repelling, or neutral.

5. Determine the type of the cycle of period 2 for the function $F(x) = -x^3$.

6. Let $D(x)$ denote the doubling function given by $2x$ if $0 \leq x < 1/2$ or $2x - 1$ if $1/2 \leq x \leq 1$. Show that all of the cycles are necessarily repelling.

7. Determine the k-value for which the logistic function undergoes a period doubling bifurcation at the fixed point.

8. Describe the bifurcations that the function $F(x) = ax + x^3$ undergoes when $a = 1$ and $a = -1$.

Exploration

Using the applets below, investigate the orbit diagram for the functions $kx(1 - x)$ and $x^2 + c$. In particular, magnify the successive regions that begin where period doubling bifurcations occur as displayed below. Do you see any similarities? Explain what you observe.

Orbit Diagram for $Cx(1-x)$:

http://math.bu.edu/DYSYS/applets/bif-dgm/Logistic.html

Orbit Diagram for $x^2 + c$:

http://math.bu.edu/DYSYS/applets/bif-dgm/Quadratic.html

Chaotic Itineraries in a Space of All Sequences

Lecture 23

Thus far we have encountered chaos in a variety of different settings, including Euler's method, the periodically forced pendulum equation, the Lorenz system, and iteration of the logistic map. In this lecture, we describe how mathematicians begin to understand chaotic behavior.

To keep things simple, let's concentrate on the logistic map $F(x) = kx(1 - x)$, where $k > 4$. Here we choose $k > 4$ so that the maximum value of F is greater than 1. When we use the computer, it appears that all orbits go to infinity. Of course that is not the case, as there are clearly 2 fixed points. Also, the graph of F^n indicates that F^n has exactly 2^n fixed points. How are these cycles arranged, and what are their periods?

The graph of F shows that all orbits of points x, with $x < 0$ or $x > 1$, go to $-\infty$. Also, there is a small open interval A_1 surrounding 1/2 and containing points that are mapped to points to the right of 1. So these points have orbits that escape from the unit interval after 1 iteration and then go to ∞. Then, by graphical analysis, there is a pair of intervals that we call A_2 that contains points that map to A_1, so these points also escape after 2 iterations. Then there are 4 intervals A_3 whose points map to A_2 and hence also escape. Next, there is a set of 2^n open intervals that contain points that escape after n iterations.

Let X be the set of points that never escape from $I = [0, 1]$. So $X = I - \cup A_n$. We want to understand the behavior of the orbits of all points in X. We see that this set is similar to the famous Cantor middle thirds set. This set is obtained as follows. Start with I. Whenever you see a closed interval, remove its open middle third. So we first remove from I the open interval (1/3, 2/3). That leaves us with 2 closed intervals, so we remove the open middle third of each, meaning (1/9, 2/9) and (7/9, 8/9). Then take this process to the limit to obtain the Cantor set.

The Cantor set is remarkable in many ways. First, it is a fractal set because it is self-similar. Second, even though it looks like it contains very few points, it actually contains exactly as many as there are in the entire unit interval! And the remaining points are not just endpoints of the removed intervals. In fact, most points in the Cantor set are not such endpoints. This is a strange set indeed.

To understand the behavior of orbits in X, we move out of the realm of the real line and quadratic functions and into a seemingly much more complicated environment, the space of all possible sequences whose entries are either 0s or 1s together with the shift map. Toward this end, we break up the unit interval I into 2 subintervals: the left half given by I_o, lying to the left of A_1, and the right half given by I_1, on the other side of A_1. (This is similar to what we observed as part of the Lorenz attractor.) Then, given any point x_0 in the unit interval, we associate an infinite sequence $S(x) = (s_0, s_1, s_2, ...)$ of 0s and 1s to x_0 via the rule $s_n = j$ if $F^n(x_0)$ lies in the interval I_j. The sequence $S(x_0)$ is called the **itinerary** of x_0.

For example, the itinerary of 0 is $S(0) = (0, 0, 0, ...)$ since 0 is a fixed point that lies in I_0. Similarly, $S(1) = (1, 0, 0, 0, ...)$ since $F(1) = 0$. Note that if $S(x_0) = (s_0, s_1, s_2, ...)$, then $S(F(x_0)) = (s_1, s_2, s_3, ...)$. That is, to obtain the itinerary of $F(x_0)$, we just drop the first entry in the itinerary of x_0. This is what we call the **shift map** σ. So we have $\sigma(s_0, s_1, s_2, ...) = (s_1, s_2, s_3, ...)$.

Now let Σ denote the set of all possible sequences of 0s and 1s modulo the identifications mentioned above; Σ is called the sequence space on 2 symbols. We say that 2 itineraries are close if they agree on the first n digits. Two itineraries are even closer together if they agree on the first $n + d$ entries. For example, the itineraries (0, 0, 0, 0, 0, ...) and (0, 0, 0, 1, 1, 1, ...) are fairly close together, but the sequences (0, 0, 0, 0, 0, ...) and (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, ...) are even closer together.

While the sequence space looks a little crazy, note that we know a lot about the fate of orbits of σ. For example, we can immediately write down all of the cycles of period n for σ. Just take any string of digits of length n and

repeat this string over and over infinitely often. The resulting sequence in Σ then lies on a cycle of period n for σ.

The amazing fact is that the dynamics of F on the unit interval are the same as that of σ on Σ. That is, each point x in I has a unique associated itinerary and vice versa: Given any itinerary in Σ, there is a unique point in I with that given itinerary. The proof of this is not too difficult and is usually given in introductory courses on iterated functions and chaos.

Because of all this, we now have a very good idea about sequences whose orbits under the shift map have very different types of behavior. Consider the sequence that begins (0, 1, 0, 0, 0, 1, 1, 0, 1, 1, ...); in other words, we first list all possible strings of 0s and 1s of length 1. Then we list all 4 possible strings of length 2, then we continue with the 8 strings of length 3, the 16 strings of length 4, and so forth. This sequence lies in Σ and so corresponds to a unique point x_0 in I. But look at what happens to the orbit of $S(x_0)$ under the shift map. If you shift this orbit a large number of times, you can arrange that this orbit comes arbitrarily close to any point in Σ! As a consequence, the orbit of x_0 must come arbitrarily close to any other point in X. This is what we call a dense orbit. These are exactly the types of orbits that seemed to occupy so much of the orbit diagram, and these are the typical orbits we see when we choose a random seed in X.

Do you see sensitive dependence on initial conditions? Given any x in X, x has the associated sequence $S(x) = (s_0, s_1, s_2, ...)$. But any nearby sequence has an itinerary that must differ from that of x at some iteration, so the 2 orbits eventually reach different intervals, I_0 and I_1. Indeed, if we just change the tail of the itinerary of x at each stage after the n^{th}, we find a nearby orbit whose eventual behavior is vastly different.

This process can be used to analyze chaos in lots of different settings. For example, consider the function on the unit circle that simply doubles the angle of a given point, that is, $\theta \rightarrow 2\theta$. We can also write this as the complex function $F(z) = z^2$. Note that $1/3 \rightarrow 2/3 \rightarrow 1/3$, so 1/3 lies on a 2-cycle; 1/7 lies on a 3-cycle; and 1/15 lies on a 4-cycle. We can do symbolic

dynamics as before by breaking the circle into 2 sets, $I_0 = [0, \pi)$ and $I_1 = [\pi, 0)$. Similar procedures to the above show that this function is chaotic on the unit circle. We could show that $\theta \to 3\theta$ is chaotic on the unit circle by breaking the circle into 3 arcs—$[0, 2\pi/3)$, $[2\pi/3, 4\pi/3)$, and $[4\pi/3, 2\pi)$—and then doing symbolic dynamics on the 3 symbols 0, 1, and 2.

Important Terms

itinerary: An infinite string consisting of digits 0 or 1 that tells how a particular orbit journeys through a pair of intervals I_0 and I_1 under iteration of a function.

shift map: The map on a sequence space that just deletes the first digit of a given sequence.

Suggested Reading

Alligood, Sauer, and Yorke, *Chaos*, chaps. 1.5–1.8.

Devaney, *A First Course in Chaotic Dynamical Systems*, chaps. 9–10.

Hirsch, Smale, and Devaney, *Differential Equations*, chap. 15.5.

Relevant Software

Iteration Applet, http://math.bu.edu/DYSYS/applets/Iteration.html

Nonlinear Web, http://math.bu.edu/DYSYS/applets/nonlinear-web.html

Problems

1. **a.** In the Cantor middle-thirds set, list the 4 intervals of length 1/27 that are removed at the third stage of the construction.

   b. What is the total length of all of the removed intervals in the first 3 stages of this construction?

   c. What is the length of all the intervals that are removed at stage 4?

   d. What is the formula for the length of all of the intervals removed at stage n?

   e. Using an infinite series, add up the lengths of all removed intervals to determine the length of the Cantor set.

2. Using symbolic dynamics, describe the itineraries of all points that eventually land on 0. How many such points are there?

3. Which points in I have orbits that eventually land on 0?

4. Give an example of a different point whose orbit fills the set X densely.

5. Give an example of a sequence whose orbit under the shift map comes arbitrarily close to the sequence (0, 0, 0, …) but never lands on this sequence.

6. Consider the shift map on the sequence space of 3 symbols—say, 0, 1, and 2. How many periodic points of (not necessarily prime) period n are there? Can you write down a sequence that has a dense orbit?

Exploration

Consider the doubling function on the interval $[0, 1)$ defined by $D(x) = 2x$ if $0 \le x < 1/2$ and $D(x) = 2x - 1$ if $1/2 \le x < 1$. (Here we exclude the point 1 from the domain.) First use a computer, a calculator, or the Iteration Applet at http://math.bu.edu/DYSYS/applets/Iteration.html to compute various orbits for the doubling function. What do you see? Is this correct behavior? Can you explain what the computer is doing?

Actually, the doubling function behaves just as chaotically as the function $4x(1 - x)$ does. Use symbolic dynamics to see this by setting $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1)$. Incidentally, there is another name for the itinerary associated to each x. What is this symbolic representation?

Conquering Chaos—Mandelbrot and Julia Sets

Lecture 24

The goal of this final lecture is to give a brief glimpse at how mathematicians are coming to grips with the compete picture of systems with chaotic behavior, and in particular, how these systems evolve as parameters vary.

The simplest example of an iterated map that exhibits a wealth of chaotic behavior is the logistic family $F(x) = kx(1 - x)$. The big question is whether we understand everything that is happening as the parameter k varies. For example, how and when do the various periodic windows arise? How do the chaotic regimes arise and change? These questions were only finally answered in the mid-1990s, and the way we found the answer to this question was to pass to the complex plane and iterate complex rather than real functions.

For historical as well as technical reasons, we will consider the quadratic function $F(z) = z^2 + c$ rather than the complex logistic maps. Here z and c are both complex numbers. The iteration of $z^2 + c$ arose in modern times thanks to the foresight of Benoit Mandelbrot. Mandelbrot used the computer to plot what is now called the Mandelbrot set (as well as the Julia sets) for these maps. The images he produced around 1980 have had a major impact on mathematics. First of all, these fractal images are amazingly intricate and beautiful, and as a consequence, they led to quite a bit of curiosity in the scientific world. More importantly, these images allowed mathematicians to use tools from complex analysis to investigate real dynamics.

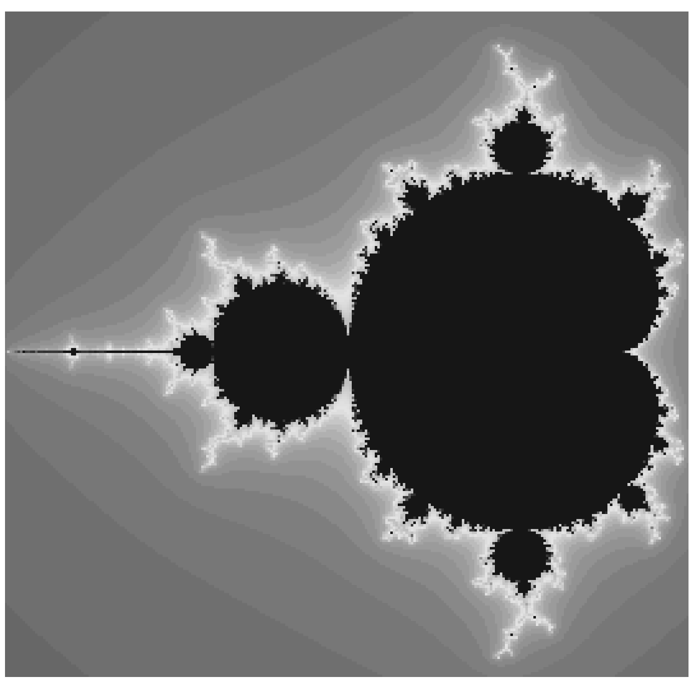
The simplest example of a complex iteration occurs when $c = 0$ (i.e., iteration of the complex function $F(z) = z^2$). It is easily seen that if $|z| < 1$, then the orbit of z tends to 0, which is an attracting fixed point. If, on the other hand, $|z| > 1$, then the orbit of z tends to ∞. Finally, if $|z| = 1$, then the orbit of z stays on the unit circle in the plane. Indeed, the behavior of F on this circle is quite chaotic. We certainly see sensitive dependence in any

neighborhood of a point on the circle since, arbitrarily close by, there are points whose orbits go far away, either to 0 or to ∞.

For the complex quadratic function $F(z) = z^2 + c$, the set of seeds z_0 for which the orbit does not tend to infinity is called the **filled Julia set**. This set is named for the French mathematician Gaston Julia, who pioneered the study of complex iteration back in the 1920s. It turns out that there are only 2 different types of filled Julia sets for $z^2 + c$: Either the filled Julia set is a connected set (just one piece), or else it is a Cantor set (infinitely many point components, sometimes called fractal dust).

As for the logistic map, the function $F(z) = z^2 + c$ has a single critical point, this time at 0. Amazingly, it is the orbit of 0 that tells us everything about the filled Julia set. For if the orbit of 0 goes to ∞, the filled Julia set is a Cantor set, but if the orbit of 0 does not tend to ∞, the filled Julia set is a connected set. This is the result that prompted Mandelbrot to plot the set of c-values in the complex plane for which the orbit of 0 does not escape (and so the filled Julia set is connected). This intricate and beautiful image is what is known as the Mandelbrot set.

Figure 24.1

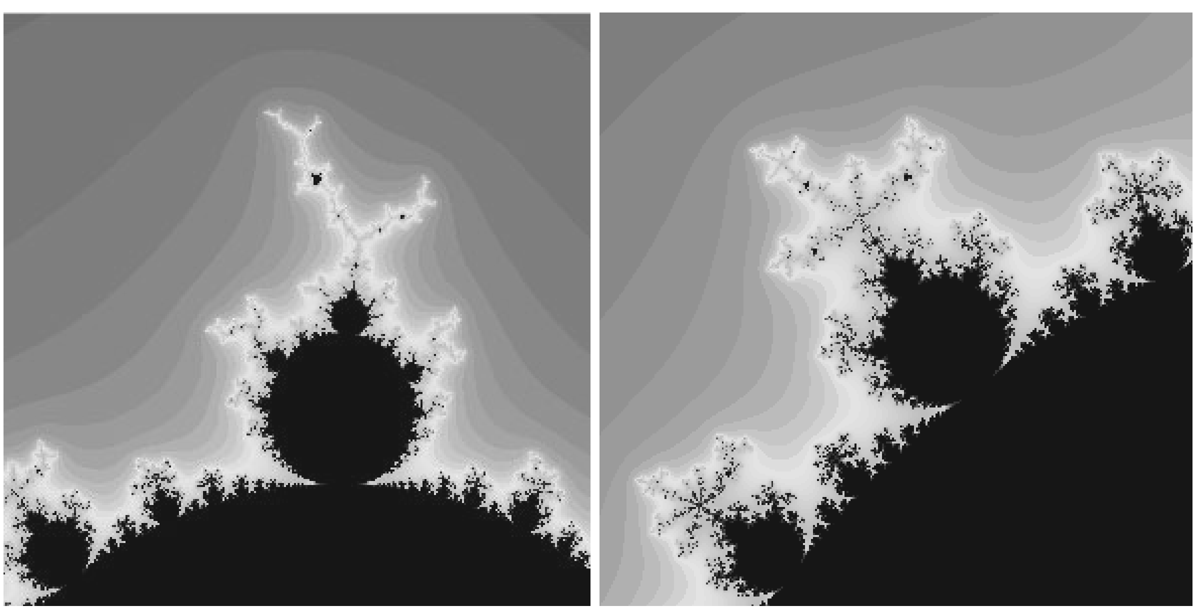


It was the geometry of the Mandelbrot set that finally allowed mathematicians to understand the real dynamical behavior of $x^2 + c$, and so also the logistic family. There are 2 reasons for this. First, the area of mathematics known as complex analysis offers many more mathematical tools to study iteration of functions like polynomials. And second, looking at objects in the plane gives us many more geometric tools to work with.

Rather than showing how the Mandelbrot set allows us to understand the behavior of $x^2 + c$ for real c-values, let's look instead at another pattern that appears. Attached to the main cardioid are infinitely many smaller bulbs. Each of these bulbs contains c-values for which the corresponding orbit of the critical point 0 tends to an attracting cycle of some period. This period is the same for all c-values drawn from a given bulb; this is the period of the bulb. So how are these periods arranged as we move around the main cardioid?

We first see that the number of spokes in the antenna attached to the bulb gives us the period of the bulb.

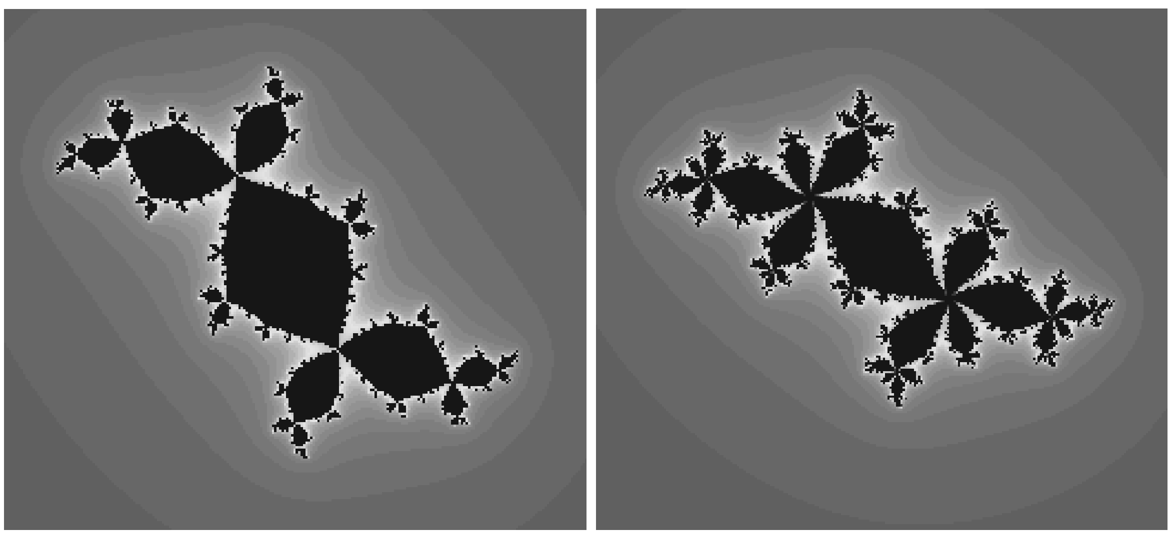
Figure 24.2



period-3 bulb period-5 bulb

We next attach a fraction to each bulb. The denominator is the period of the bulb. The numerator can be defined in 3 different ways. (1) The location of the smallest spoke in the antenna relative to the principal spoke (the spoke that connects to the bulb itself) going in the counterclockwise direction gives us the numerator. Above are the 1/3 and 2/5 bulbs. (2) If we plot a filled Julia set corresponding to a *c*-value from the bulb, the smallest ear hanging off the central portion of the set, again counted in the counterclockwise direction, also determines the numerator. Below are filled Julia sets drawn from the 1/3 and 2/5 bulbs. (3) The rotation number of the attracting cycle also gives the numerator.

Figure 24.3



It turns out that the bulbs are arranged in the exact order of the rational numbers. So we see a lot of connections between the geometry of the Mandelbrot set and Julia sets and the dynamical behavior. Proving these facts is not that easy, but this nonetheless shows how mathematicians use tools from diverse areas of mathematics to understand the behavior of complicated systems of differential equations.

Important Term

filled Julia set: The set of all possible seeds whose orbits do not go to infinity under iteration of a complex function.

Suggested Reading

Devaney, *A First Course in Chaotic Dynamical Systems*, chap. 17.

———, *The Mandelbrot and Julia Sets*.

Mandelbrot, *Fractals and Chaos*, chap. 1.

Peitgen, Jurgens, and Saupe, *Chaos and Fractals*, chaps. 13–14.

Relevant Software

The Quadratic Map, http://math.bu.edu/DYSYS/applets/Quadr.html

The Mandelbrot Set Iterator, http://math.bu.edu/DYSYS/applets/M-setIteration.html

Orbit Diagram for $x^2 + c$, http://math.bu.edu/DYSYS/applets/bif-dgm/Quadratic.html

Problems

1. Compute the orbit of i under $F(z) = z^2$ and describe its fate.

2. Compute the orbit of $2i$ under $F(z) = z^2$ and describe its fate.

3. Compute the orbit of $i/2$ under $F(z) = z^2$ and describe its fate.

4. Which orbits of $F(z) = z^2$ tend to infinity?

5. Which orbits of $F(z) = z^2$ tend to the origin?

6. For the quadratic function $x^2 + c$, we saw that there were no fixed points on the real line when $c > 1/4$. What happens in the complex plane?

7. What is the fate of the orbit of 0 under $z^2 + i$? Is the Julia set connected?

8. Use the computer to investigate what happens to the filled Julia sets for c-values along the path $c = -3/4 + iA$ (where A is a parameter).

9. Use the computer to look at the portion of the Mandelbrot set that lies along the real axis. What object in the Mandelbrot set corresponds to the windows in the orbit diagram for $x^2 + c$?

10. Discuss the bifurcation that occurs at $c = 1/4$ along the real axis, but now from the complex point of view.

Exploration

Consider the bulbs hanging off the period-2 and period-3 bulbs attached to the main cardioid of the Mandelbrot set. How are the periods of these bulbs arranged? You may use The Mandelbrot Set Iterator software located at http://math.bu.edu/DYSYS/applets/M-setIteration.html to find these periods.

Solutions

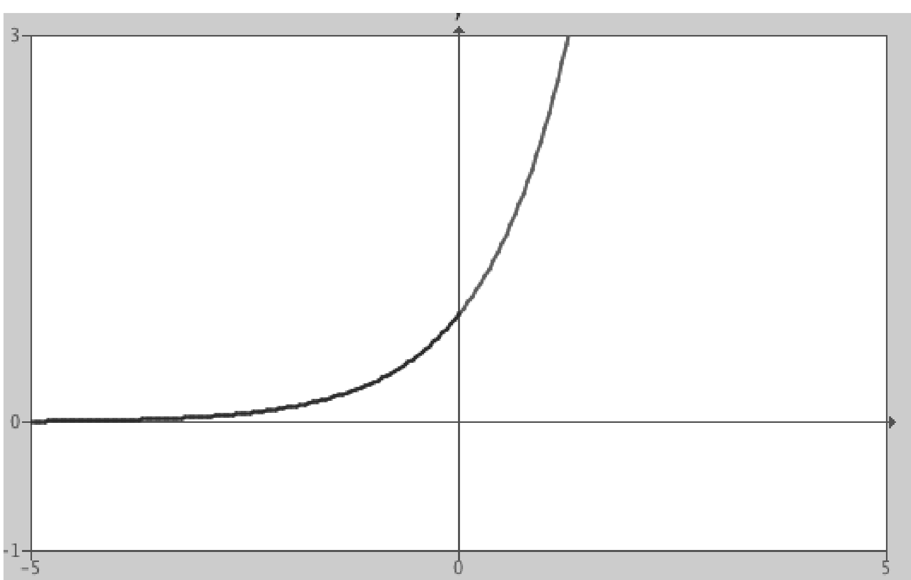
Lecture 1

1. **a.** $y'(t) = 3t^2 + e^t$.

   b. $y''(t) = 6t + e^t$.

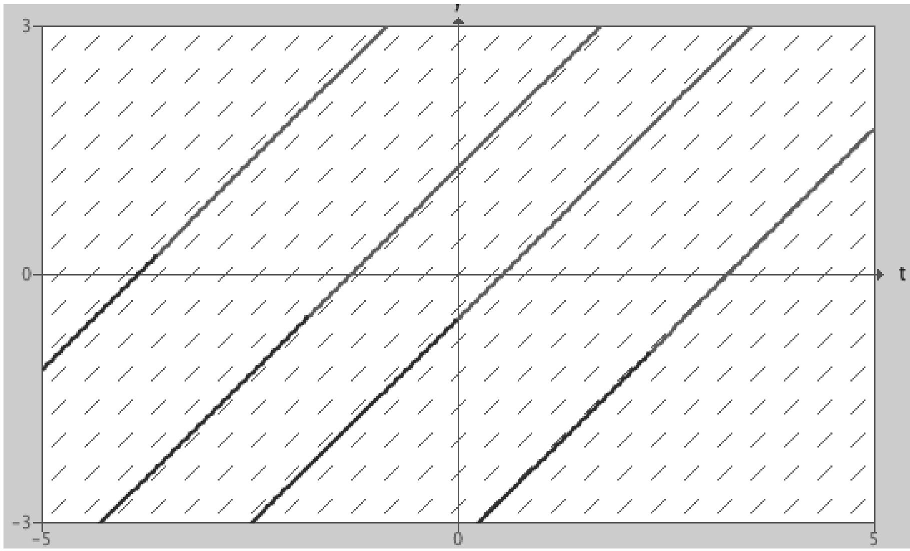
   c. $f(t) = t^4/4 + e^t$.

   d. The graph is always positive and increasing. Moreover, e^t tends to 0, and t tends to $-\infty$ and to ∞ as t tends to positive infinity.



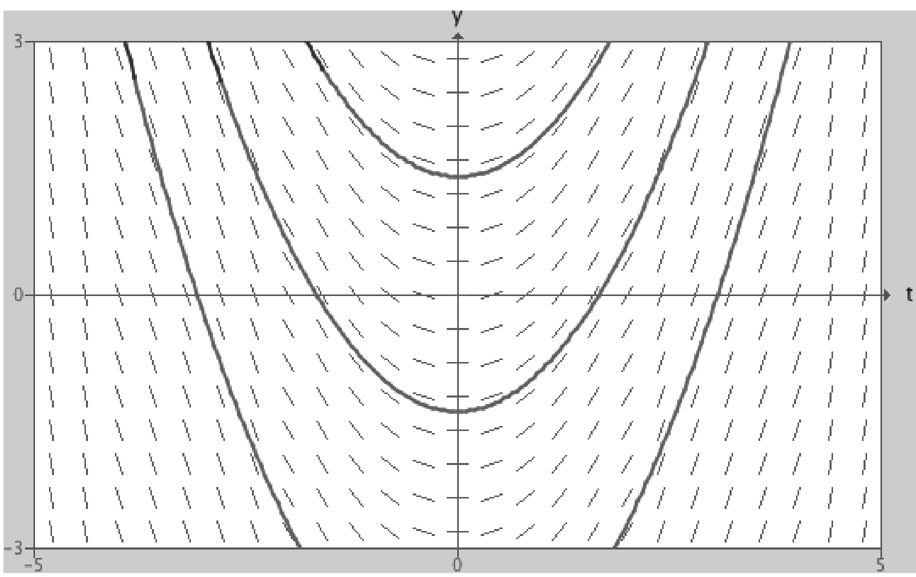
   e. $t = 0$.

2.



3. Solutions are of the form $y(t) = t + C$, where C is a constant.

4.



Solutions are of the form $y(t) = t^2/2 + C$.

5. All solutions tend to 0 since $y' < 0$ when $y > 0$. As before, $y(t) = e^{kt}$, but now $k < 0$.

6. The general solution here would by $y(t) = t^2/2$ + constant.

7. Equilibria occur at $y = 1$ and $y = -1$. We have $y' > 0$ if $y > 1$ and $y < -1$, whereas $y' < 0$ if $-1 < y < 1$. So solutions tend to infinity if $y > 1$ and to -1 if $y < 1$.

8. A solution to this differential equation for any constant is $y(t)$ = constant. So solutions never move; they stay put.

9. First write $y'' = g/m$. Then we must have $y' = (g/m)t + A$ for some constant A. And then $y(t) = (1/2)(g/m)t^2 + At + B$, where B is any other constant.

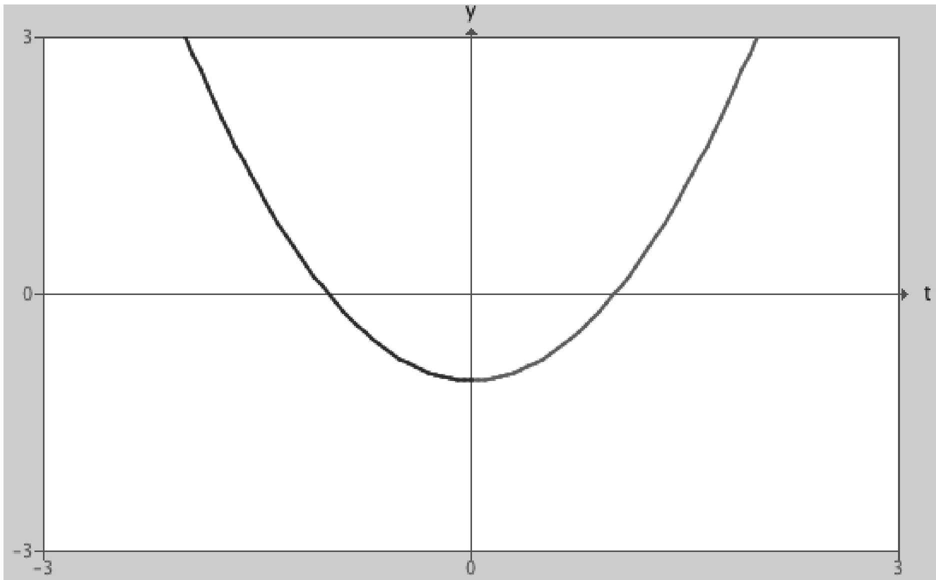
Lecture 2

1. **a.** $t^3/3 + t^2/2$.

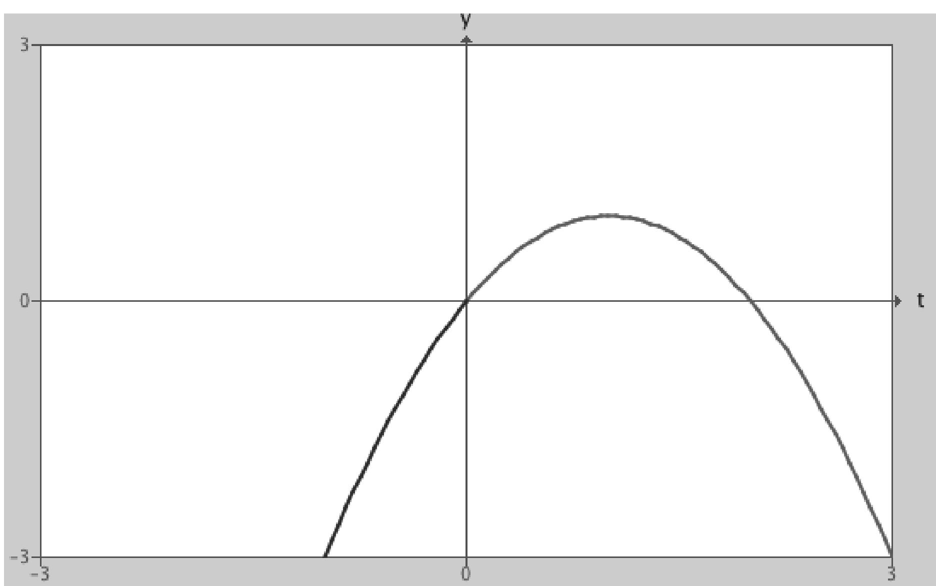
b. $y = 0$, 2, and -2 are solutions.

c. $y(t) = 0$ when $t = 0$, 1, and -1. $y(t) > 0$ if $t > 1$ or $-1 < t < 0$.

d. The graph of $y(t) = t^2 - 1$ is a parabola opening upward and crossing the (horizontal) t-axis at $t = 1$ and $t = -1$.



e. This graph crosses the horizontal t-axis at $t = 0$ and $t = 2$. The graph is a parabola opening downward since $y(t)$ tends to $-\infty$ as t approaches $\pm\infty$.



2.

3.

4. The only equilibrium point is 0.

5. There are no equilibria for this differential equation.

6. For n even, all solutions with $y > 0$ tend to ∞ while all solutions with $y < 0$ tend to 0. When n is odd, again all solutions with $y > 0$ tend to ∞ while all solutions with $y < 0$ now tend to $-\infty$. So the answer does depend on n.

7. There are equilibria at -1 and 0. If $y > 0$, then $y' > 0$, so solutions go to ∞. If $-1 < y < 0$, then $y' > 0$, so solutions increase to 0. If $y < -1$, then $y' < 0$, so solutions tend to $-\infty$.

8. The function $\sin(y)$ has equilibria at $n\pi$ for each integer n. Between 0 and π, $\sin(y)$ is positive, so solutions increase to $y = \pi$. Between π and 2π, $\sin(y)$ is negative, so solutions decrease to $y = \pi$. Similar behavior occurs in other intervals of length 2π; solutions always tend to the equilibria at $y = n\pi$ when n is odd and away from the equilibria at $y = n\pi$ when n is even.

9. What about $y' = \sin^2(\pi y)$? y' is always positive except at the integers where $y' = 0$.

10. One example would be $y' = |y(1 - y)|$ since $y' > 0$ everywhere except $y = 0$ and $y = 1$, but of course there are many others, like $y' = y^2(1 - y)^2$.

Lecture 3

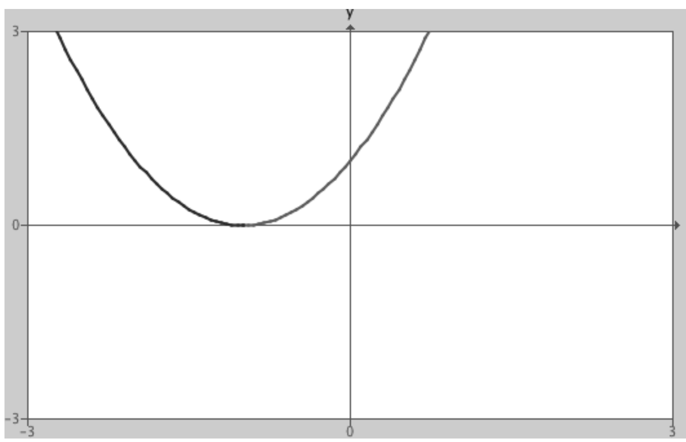
1. **a.** $y'(t) = 2t$, so $y(t)$ increases when $t > 0$.

   b. $y(t)$ decreases when $t < 0$.

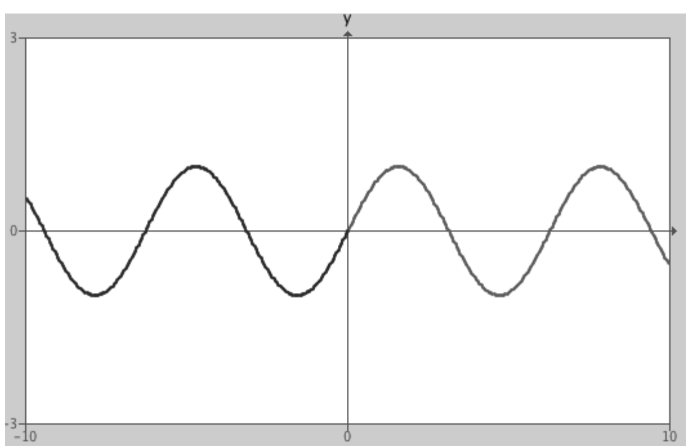
   c. $y(t) = (t + 1)^2$ so the only root is $t = -1$.

   d. $y' = 2t + 2$, so this function increases when $t > -1$.

   e.



   f.



   g. $-3\sin(3t + 4)$.

2. The only equilibrium point is $y = -1$, which is a source.

3. The only equilibrium point is $y = 0$, which is a node.

4.

5. Equilibria at $y = 1$ (source) and $y = -1$ (sink).

6. The only equilibrium point is at $y = 1$, and there we have that the derivative of $y^3 - 1$ is $3y^2$. At 1 we get $y' = 3$, so 1 is a source.

7. Technically, the existence and uniqueness theorem does not apply when $y = 0$ since the function $|y|$ is not differentiable there. However, we have an equilibrium solution $y = 0$ there, and all other solutions are given by $y(t) = Ce^t$ when $C > 0$ or Ce^{-t} when $C < 0$, so we do have existence and uniqueness at $y = 0$.

8. The equilibrium points are given by $\pm A^{1/2}$ when $A > 0$ and 0 when $A = 0$. There are no equilibrium points when $A < 0$. Since the derivative of $y^2 - A$ is $2y$, we have that $+A^{1/2}$ is a source while $-A^{1/2}$ is a sink. When $A = 0$, the equilibrium point at 0 is a node.

9. The equilibria are at 0 and 1 and the derivative is $A(1 - 2y)$. So for $y = 0$, this point is a source when $A > 0$ and a sink when $A < 0$. For $y = 1$, we have a sink when $A > 0$ and a source when $A < 0$. When $A = 0$, all points are equilibrium points, so they are all nodes.

10. This equation has an equilibrium point at $y = -1$. For $y < -1$, $y' < 0$, so these solutions tend to $-\infty$. For $-1 < y < 1$, solutions now increase until they hit $y = 1$, where the slope becomes infinite, so the solutions stop there. If $y > 1$, then $y' < 0$, so solutions decrease until they hit $y = 1$, when again the slope becomes infinite and solutions stop.

Lecture 4

1. The only equilibrium point is at $y = A$, and this point is a source.

2. No.

3. There are no equilibrium points when A is nonzero and infinitely many when $A = 0$; every y-value is an equilibrium point when $A = 0$.

4. A bifurcation occurs when $A = 0$.

5. The equilibria occur at $y = 0$ and $y = -1/A$ (as long as A is nonzero).

6. When $A > 0$, there is an equilibrium point at $y = 0$ that is a source. When $A < 0$, the equilibrium point at $y = 0$ is a sink. But when $A = 0$, all points are equilibrium points, so we have a bifurcation at $A = 0$.

7. We have equilibria at $y = 0$ and $y = A$ for each A. When $A = 0$, there is a single equilibrium point at $y = 0$, a node, with all other solutions decreasing. When $A > 0$, the equilibrium point at $y = A$ is a sink while 0 is a source. The opposite occurs when $A < 0$.

8. Only at those that are nodes.

9. When $B = 0$, we have a similar situation to that in problem 7. When $B > 0$, we always have a pair of equilibria for each A, given by

$$\frac{A \pm \sqrt{A^2 + 4B}}{2}.$$

When $B < 0$, these 2 equilibria only exist if $A^2 + 4B \geq 0$ (i.e., for $B \geq -A^2/4$). When $B = -A^2/4$, there is a single equilibrium point at $A/2$. When there are 2 equilibria, the larger one is a sink and the other is a source.

10. When $B = 0$, we have a single equilibrium point at $y = 0$ when $A \leq 0$. This equilibrium point is a sink. There are 3 equilibria if $A > 0$: one at $y = 0$, a source; and 2 sinks at $y = \pm A^{1/2}$. When $B \neq 0$, the graph of $F(y) = B + Ay - y^3$ has derivative equal to 0 when $3y^2 = A$. Therefore if $A < 0$, the graph of $F(y)$ is strictly decreasing, so there is always a single equilibrium point that is a sink. If $A > 0$, there are now 2 places where $F'(y) = 0$, at $y = \pm(A/3)^{1/2}$. $F(y)$ has a local minimum at the point $y = -(A/3)^{1/2}$ and a local maximum at $y = +(A/3)^{1/2}$. For B-values for which the value of F at the local minimum is greater than 0, the graph of F shows that there is only one equilibrium point, a sink. Similarly, there is only one equilibrium point if the value of F at the local maximum is less than 0, again a sink. But if the local minimum is less than 0 and the local maximum is greater than 0, then there are 3 equilibrium points. The largest and smallest equilibria are sinks, and the other is a source. When either the local maximum or local minimum value is 0, then a sink and a source merge to become a node.

Lecture 5

1. **a.** $t^2/2 + 5t + C$.

   b. $t^3/3 + t^2 + t + C$.

   c. $-t^{-1} + C$.

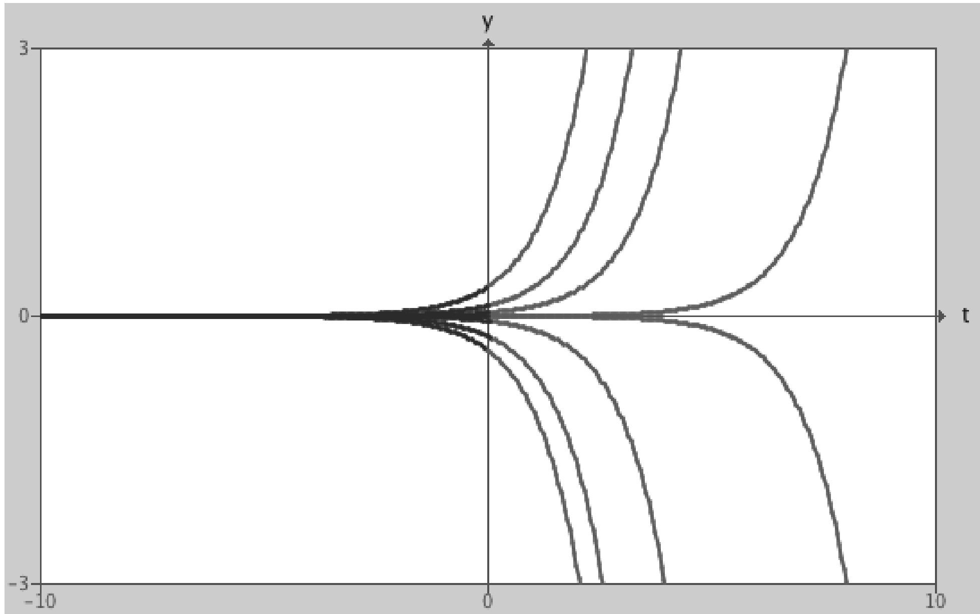
   d. $-e^{-t} + C$.

   e. C.

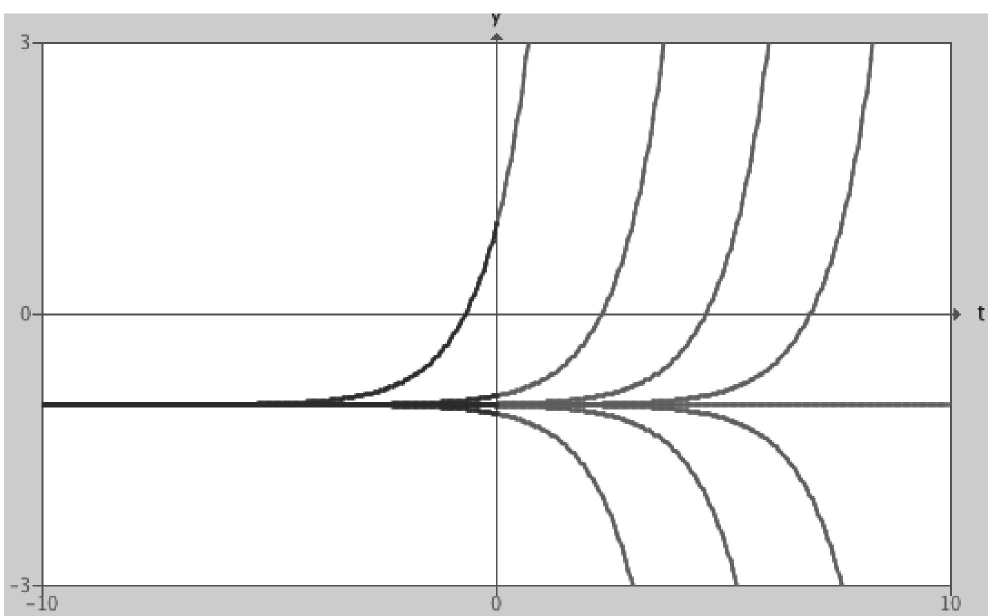
2. $t = e^4$.

3. $t = \ln(4)$.

4. $y(t) = Ce^t$.



5. $y(t) = Ce^t - 1$.



6. We have the supposed solution

$$y(t) = \frac{De^t}{1 + De^t}.$$

First use the quotient rule to compute that

$$y'(t) = \frac{De^t}{\left(1 + De^t\right)^2}.$$

Next compute

$$y(1 - y) = \frac{De^t}{1 + De^t}\left(1 - \frac{De^t}{1 + De^t}\right) = \frac{De^t}{\left(1 + De^t\right)^2}$$

so we see that this is indeed a solution.

7. Separating and integrating, we find

$$\frac{-1}{y} = \int \frac{dy}{y^2} = \int dt = t + c$$

so that $y(t) = -1/(t + c)$. This is not the general solution, since when $y(0) = 0$, we have $0 = y(0) = -1/c$, or multiplying through by c, we find that $0 = -1$. Is that true? Obviously not. What's wrong here? The answer is that the solution to the initial value problem $y(0) = 0$ is just the constant function $y(t) = 0$ (i.e., an equilibrium solution). So the solutions $-1/(t + c)$ is not quite the general solution; we must also add in the solution $y(t) = 0$ to solve all initial value problems.

8. The differential equation here is $y' = k(y - 80)$. This equation is separable, so we have

$$\ln|y-80| = \int \frac{dy}{y-80} = \int k\,dt = kt + c.$$

Assuming $y(t) > 80$, we find by exponentiating both sides that $y(t) = 80 + e^{kt + c} = 80 + De^{kt}$. Since $y(0) = 200$, we also have $200 = y(0) = 80 + De^0$ so that $D = 120$. So our solution so far is $y(t) = 80 + 120e^{kt}$. But we also have $y(1) = 180$, so that $180 = 80 + 120e^k$. Therefore $5/6 = e^k$ or $k = \ln(5/6)$. Thus the full solution is $y(t) = 80 + 120e^{(\ln(5/6))t} = 80 + 120(5/6)^t$.

9. First off, the solution satisfying the initial condition $y(0) = 1$ is clearly the constant solution $y(t) = 1$ (i.e., one of the 2 equilibrium solutions). For the other solutions we can again separate and integrate

$$\int \frac{dy}{1-y^2} = \int dt = t + c.$$

To integrate $1/(1-y^2)$, note that we can break up this fraction into

$$\frac{1}{1-y^2} = \frac{1/2}{1+y} + \frac{1/2}{1-y}.$$

First assume $y(0) = 0$. Then our solution lies below the equilibrium solution $y = 1$, so we may integrate to find

$$\int \left(\frac{1/2}{1+y} + \frac{1/2}{1-y} \right) dy = \tfrac{1}{2}\ln(1+y) - \tfrac{1}{2}\ln(1-y) = t + c.$$

Exponentiating both sides then yields

$$\sqrt{\frac{1+y}{1-y}} = ke^{t}$$

or

$$\left(\frac{1+y}{1-y}\right) = De^{2t}.$$

Solving for y then yields

$$y(t) = \frac{De^{2t}-1}{1+De^{2t}},$$

which satisfies $y(0) = 0$ when $D = 1$. Similar calculations show that this expression also solves the initial value problem $y(0) = 2$ when $D = -3$.

10. Notice that our solution above includes the equilibrium solution $y(t) = -1$ when $D = 0$. Also, solving the initial value problem $y(0) = A$ shows that $A = (D - 1)/(1 + D)$ so that $D = -(A + 1)/(A - 1)$, which is fine as long as A is not equal to 1. But we know the solution that solve the initial value problem $y(0) = 1$; it is our equilibrium solution $y(t) = 1$, so we must add this solution to the other solution to get the full general solution.

Lecture 6

1. **a.** 3/2.

   b. $(y_1 - y_0)/(t_1 - t_0)$.

   c. $y = -t + 1$.

   d. $y = 0$.

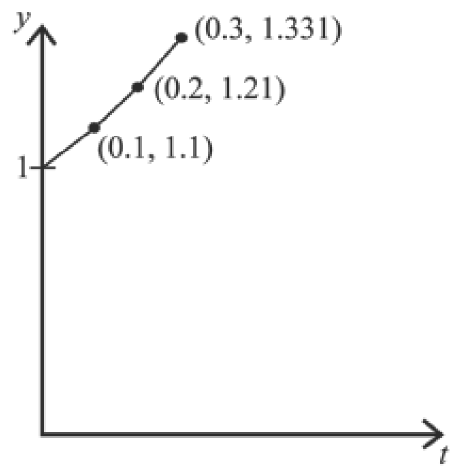
   e. $t = 1$.

2. $y' = 2t$, so the slope at $t = 1$ is 2. So our equation so far is $y = 2t + B$. To determine B, we know that the point (1, 1) lies on this straight line. So we have $1 = 2 + B$, so $B = -1$. Thus the equation is $y = 2t - 1$.

3. $t_1 = 0.1$, and $y_1 = 1.1$.

4. $t_2 = 0.2$, $y_2 = 1.21$, $t_3 = 0.3$, $y_1 = 1.331$.

5.



6. **a.** Clearly, this solution is just e^t.

b. Using a spreadsheet, I calculate that $y(1) = 2.593743$ when the step size is 0.1.

c. With step size 0.05, I find $y(1) = 2.653298$ and with step size .01 I find $y(1) = 2.704814$ (your approximations may differ slightly depending on the software you use). Clearly, we are getting better approximations of the actual solution.

d. Using the value of $y(1) = 2.718281$, the error when the step size is 0.1 is 0.124538; when the step size is 0.05, the error is 0.064983; when the step size is 0.01, the error is 0.013467. So when we cut the step size in half by going from step size 0.1 to 0.05, the error decreases by approximately one half. And when we decrease the step size by 1/5 when we go from step size 0.05 to 0.01, the error also decreases by approximately 1/5.

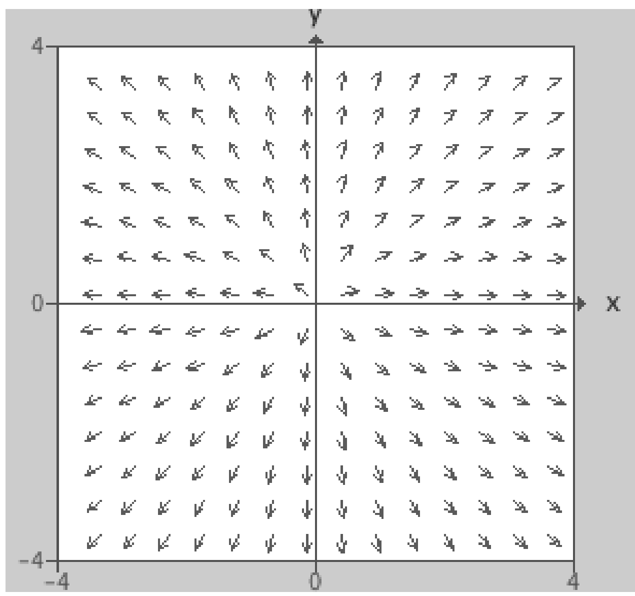
7. Any solution that starts above the equilibrium point at $y = -1$ tends toward the value $y = 1$ where the slope field has infinite slope. Here the numerical method "goes crazy," and we again see chaotic behavior.

Lecture 7

1. Only the origin is an equilibrium point.

2. All points along the line $y = -x$ are equilibrium points.

3. **a.**



b. It appears that all solutions (except the equilibrium point at the origin) move away from the origin along a straight line.

c. Since this system decouples, solutions are of the form

$$x(t) = k_1 e^t$$

$$y(t) = k_2 e^t.$$

4. $y' = v$

$v' = y$

5. **a.** The direction field is always tangent to the circles that are centered at the origin and point in the clockwise direction.

b. One solution is $x(t) = \cos(t)$ and $y(t) = -\sin(t)$. Another is $x(t) = \sin(t)$ and $y(t) = \cos(t)$. Any constant times each of these is also a solution.

6. **a.** The vector field is horizontal along the x-axis (pointing to the right if $0 < x < 1$ and to the left if $x < 0$ or $x > 1$). The vector field is vertical along the y-axis and always points toward the origin. At any point off the axes, the x and y directions of the vector field are the same as the corresponding directions on the axes.

b. The equilibria are (0, 0) and (1, 0).

c. On the x-axis, when $x > 0$, all solutions tend to 1, and when $x < 0$, all solutions tend to $-\infty$. Meanwhile, on the y-axis all solutions tend to 0. So if $x > 0$, the solution tends to (1, 0) and if $x < 0$, the solution tends off to ∞ to the left.

Lecture 8

1. $y'' + 3y' + 2y = 0.$

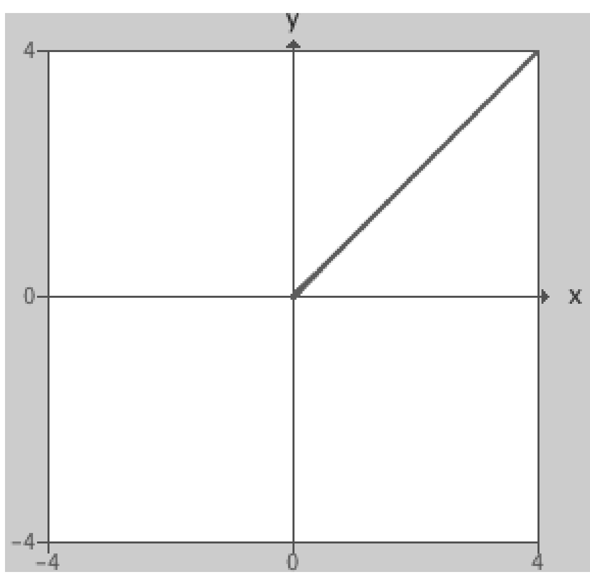
2. $y' = v$

$v' = -2y - 3v.$

3. The only equilibrium point lies at the origin.

4. $-4\sin(2t)$ and $-4\cos(2t)$.

5.



6. We need to solve $k_1 + k_2 = A$, $-k_1 - 2k_2 = B$ for any given values of A and B. Adding these equations yields $-k_2 = A + B$, so $k_2 = -A - B$. Then the first equation implies that $k_1 = 2A + B$.

7. **a.** The characteristic equation is $(s + 3)(s + 2)$, and the general solution is $k_1e^{-3t} + k_2e^{-2t}$.

b. We must solve $k_1 + k_2 = 0$, $-3k_1 - 2k_2 = 1$, which yields $k_1 = -1$ and $k_2 = 1$.

c. The graph of $y(t)$ increases at first, then reaches a maximum, and then slowly decreases to 0.

d. Initially the mass moves upward, but then it turns around and glides directly back to its rest position.

Lecture 9

1. $2e^{2t}\cos(3t) - 3e^{2t}\sin(3t)$.

2. $e^{2t}(\cos(3t) + i\sin(3t))$.

3. $\dfrac{-b \pm \sqrt{b^2 - 4k}}{2}$.

4. In the clockwise direction.

5. The characteristic equation is $s^2 + 5s + 6 = (s + 3)(s + 2)$ with roots -2 and -3, so this system is overdamped.

6. L'Hopital's rule says that the limit as $t \to \infty$ of t/e^t is the same as the limit of the quotient of the derivatives (i.e., $1/e^t$). This quotient tends to 0 as $t \to \infty$.

7. The roots of the characteristic equation are $-1 \pm I$, so the general solution is $k_1e^{-t}\cos(t) + k_2e^{-t}\sin(t)$.

8. The solution that satisfies $y(0) = 0$ and $y'(0) = 1$ is $y(t) = e^{-t}\sin(t)$. The derivative is $y'(t) = -e^{-t}\sin(t) + e^{-t}\cos(t)$. This derivative vanishes when $\sin(t) = \cos(t)$, so the first positive t-value where $y'(t) = 0$ is when $t = \pi/4$. Then we have $y(\pi/4) = e^{-\pi/4}\sin(\pi/4) = e^{(-\pi/4)/\sqrt{2}}$.

9. The characteristic equation is $s^2 + bs + 1$, whose roots are

$$\frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

So the system is undamped when $b = 0$, underdamped when $0 < b < 2$, critically damped when $b = 2$, and overdamped when $b > 2$.

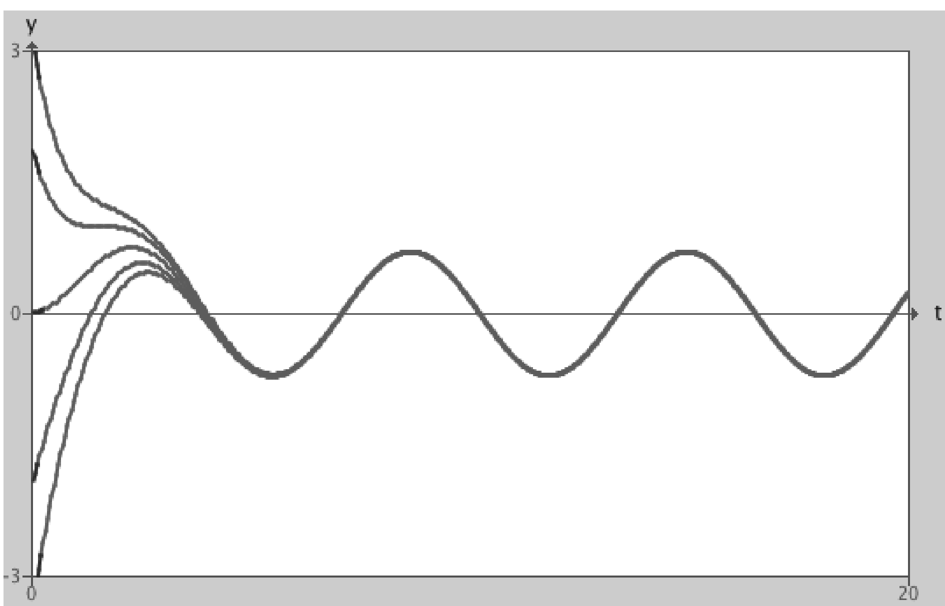
10. Clearly $y(t) = 1$ is a constant solution to the nonhomogeneous equation. So the general solution, as in the first-order case, is $k_1\cos(t) + k_2\sin(t) + 1$. That is, the mass just oscillates about a point 1 unit removed from the natural equilibrium position.

Lecture 10

1. **a.** $y(t) = ke^{-t} + (1/2)\sin(t) - (1/2)\cos(t)$.

b. All solutions tend to the periodic solution $(1/2)\sin(t) - (1/2)\cos(t)$.

c.



d. The solutions are now $y(t) = ke^{t} - (1/2)\sin(t) - (1/2)\cos(t)$, so solutions no longer tend to the periodic solution given by $-(1/2)\sin(t) - (1/2)\cos(t)$ as t increases.

e. As time tends to $-\infty$, solutions now tend to the periodic solution.

2. Make the guess of $A\cos(t) + B\sin(t)$ to find the solution with $A = -1/2$ and $B = 1/2$.

3. The general solution is $k_1\cos(t) + k_2\sin(t) + (1/2)e^{-t}$.

4. When $w = \sqrt{3}$, the system is in resonance.

5. In order for $\cos(wt) + \cos(t)$ to be periodic, we must find a constant A for which

$$\cos(w(t + A)) + \cos(t + A) = \cos(wt) + \cos(t)$$

for all t-values. In particular, this must be true when $t = 0$. But then we have $\cos(wA) + \cos(A) = 2$. This implies that we must have $\cos(wA) = 1 = \cos(A)$. Therefore we need both wA and A to be integer multiples of 2π. So we must have $wA = 2n\pi$ and $A = 2m\pi$ for some integers n and m, so $w = n/m$. That is, w must be a rational number.

6. When w is a rational number, solutions are then periodic in t.

Lecture 11

1. $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$.

2. **a.** $Y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y$.

b. Only the origin.

3. No.

4. **a.** Yes.

b. No.

5. **a.** $Y' = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} Y$.

b. The first solution is $x(t) = k_1 e^t$. Then we must solve $y' = 2y + k_1 e^t$. The equation $y' = 2y$ has general solution $y(t) = k_2 e^{2t}$. Therefore for the nonhomogeneous equation, we guess Ce^t so that $C = -k_1$. So our solutions are $x(t) = k_1 e^t$ and $y(t) = k_2 e^{2t} - k_1 e^t$.

c. $Y(t) = \begin{pmatrix} k_1 \exp(t) \\ k_2 \exp(2t) - k_1 \exp(t) \end{pmatrix} = k_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

d. We must be able to solve $x(0) = A$ and $y(0) = B$ for any A and B. The first equation gives $A = k_1$, and the second then says $k_2 - A = B$ or $k_2 = A + B$, so this is indeed the general solution.

e. When $k_1 = 0$, we have a straight line solution along the y-axis moving away from the origin. When $k_2 = 0$, we find a straight line solution along the line $y = -x$ again moving away from the origin. All other solutions also tend away from the origin.

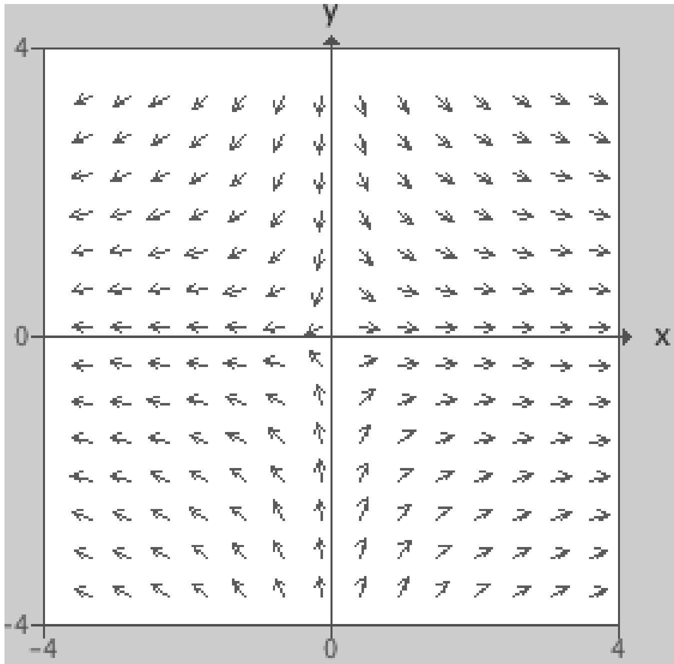
Lecture 12

1. The determinant is −5.

2. Since the determinant is nonzero, the only equilibrium point is at the origin.

3. The trace is 4, and the determinant is 3.

4. **a.**



b. It appears there are straight line solutions along the x- and y-axes.

5. 2 and -1.

6. The eigenvalues are just a and b, since the characteristic equation is $(a - \lambda)(b - \lambda) = 0$.

7. The eigenvalues are both 0, since this is an upper triangular matrix. However, every nonzero vector is an eigenvector since multiplying this vector by the matrix yields (0, 0): that is, 0 times the given vector.

8. The characteristic equation here is $\lambda^2 - (1 + 3\sqrt{2})\lambda$. So the eigenvalues are 0 and $1 + 3\sqrt{2}$. The eigenvector corresponding to the eigenvalue 0 is given by any nonzero solution of the equation

$x + 3y = 0$, so for instance, the vector $(1, -1/3)$ is one such eigenvector. The eigenvector corresponding to the eigenvalue $1 + 3\sqrt{2}$ is given by solving the equation $\sqrt{2}\,x - y = 0$. So one eigenvector is $(1, \sqrt{2})$.

9. The characteristic equation here is $\lambda^2 - 6\lambda + 8 = 0$, so the eigenvalues are 4 and 2. An eigenvector corresponding to the eigenvalue 4 is any nonzero vector of the form $y = x$, so $(1, 1)$ is an eigenvector for this eigenvalue. An eigenvector corresponding to the eigenvalue 2 is any nonzero vector satisfying $y = -x$, so $(1, -1)$ is one such eigenvector. Thus the general solution is

$$k_1 e^{4t}\begin{pmatrix}1\\1\end{pmatrix} + k_2 e^{2t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

The solutions with k_1 or k_2 equal to 0 are straight line solutions moving away from the origin and lying along the lines $y = x$ and $y = -x$. All other solutions also move away from the origin tangentially to the straight line solutions along $y = -x$.

10. The solution satisfying this initial condition is found by solving the system of equations

$$k_1 + k_2 = 1$$

$$k_1 - k_2 = 0$$

so $k_1 = k_2 = 1/2$ yield this solution.

Lecture 13

1. **a.** The characteristic equation is $\lambda^2 + 1 = 0$.

b. The eigenvalues are $\pm i$.

c. One eigenvector associated to the eigenvalue i is

$$\begin{pmatrix} 5 \\ i-2 \end{pmatrix},$$

and an eigenvector associated to $-i$ is

$$\begin{pmatrix} 5 \\ -i-2 \end{pmatrix}.$$

2. **a.** The eigenvalues are 0 and -1.

b. One eigenvector corresponding to 0 is (1, 0) and corresponding to -1 is $(-1, 1)$.

3. As a system, $y'' + by' + ky = 0$ may be written

$y' = v$, $v' = -ky - bv$ or

$$Y' = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} Y.$$

The eigenvalues are roots of $\lambda^2 + b\lambda + k = 0$ (which is the same characteristic equation that we saw for the second order equation), and so are given by

$$\frac{-b \pm \sqrt{b^2 - 4k}}{2}.$$

4. **a.** The characteristic equation is $\lambda^2 - 2a\lambda + a^2 + b^2 = 0$, which has roots given by $a + ib$ and $a - ib$. The eigenvector for $a + ib$ is found by solving

$$-ibx + by = 0$$

$$-bx - iby = 0,$$

so $y = ix$. One complex eigenvector is therefore $(1, i)$. For the eigenvalue $a - ib$, the equations are

$$ibx + by = 0$$

$$-bx + iby = 0,$$

so $y = -ix$. One eigenvector in this case is $(1, -i)$.

b. For the eigenvalues $a \pm ib$, we have a spiral source if $a > 0$, a spiral sink if $a < 0$, and a center if $a = 0$. If $a = b = 0$, we have real and repeated 0 eigenvalues.

5. **a.** The characteristic equation is

$$\lambda^2 - (1+3\sqrt{2})\lambda = 0,$$

so the eigenvalues are 0 and $1 + 3\sqrt{2}$. The eigenvector corresponding to 0 is given by $x + 3y = 0$ or $(3, -1)$. The eigenvector for $1 + 3\sqrt{2}$ is given by solving $\sqrt{2}\,x - y = 0$ or $(1, \sqrt{2})$. In the phase plane, we have a straight line of equilibrium points along the line $x + 3y = 0$. All other solutions lie on straight lines with slope $\sqrt{2}$, and these solutions tend directly away from the single equilibrium point on this line as time goes forward.

b. Given the general solution

$$k_1\begin{pmatrix}3\\-1\end{pmatrix} + k_2 e^{(1+3\sqrt{2})t}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix},$$

we must solve

$$3k_1 + k_2 = 1$$

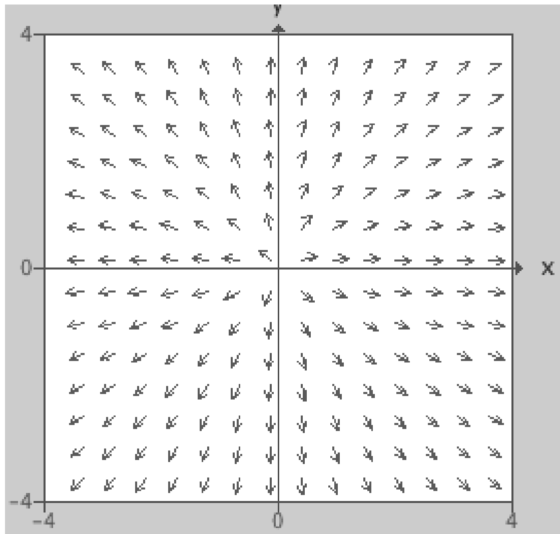
$$-k_1 + \sqrt{2}\,k_2 = 0$$

so we have $\sqrt{2}\,k_2 = k_1$. Then equation 1 implies $3\sqrt{2}\,k_2 + k_2 = 1$ so that $k_2 = 1/(1 + 3\sqrt{2})$ and so $k_1 = \sqrt{2}/(1 + 3\sqrt{2})$. So the solution is given by

$$\frac{\sqrt{2}}{1+3\sqrt{2}}\begin{pmatrix}3\\-1\end{pmatrix} + \frac{e^{(1+3\sqrt{2})t}}{1+3\sqrt{2}}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}.$$

Lecture 14

1. a.



b. The eigenvalues are 2 repeated.

c. Every nonzero vector is an eigenvector.

d. One of many possible forms for the general solution is

$$k_1 e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + k_2 e^{2t}\begin{pmatrix}0\\1\end{pmatrix}.$$

e. All nonzero solutions move away from the origin along straight lines.

2. We have repeated eigenvalues given by -1. An eigenvector corresponding to -1 is given by solving $x + 0y = 0$, so one eigenvector is $(0, 1)$. Next we solve the equations $0x + 0y = 0$ and $x + 0y = 1$ to find the special vector $(1, 0)$. Then the general solution is

$$k_1 e^{-t}\begin{pmatrix}0\\1\end{pmatrix} + k_2 t e^{-t}\begin{pmatrix}0\\1\end{pmatrix} + k_2 e^{-t}\begin{pmatrix}1\\0\end{pmatrix}.$$

3. We could solve this using the eigenvalue/eigenvector method as in the previous question, but it is much easier to proceed as follows. Our equations read $x' = 0$ and $y' = x$. So we have $x(t) = k_1$, and then integration yields $y(t) = k_1 t + k_2$.

4. **a.** We have $T = a$ and $D = -a$, so the path lies along the line $D = -T$ in the TD-plane. When $a = -4$ and $a = 0$, this line meets the repeated eigenvalue parabola $T^2 - 4D = 0$. When $a < -4$, we are in the real sink region. For $-4 < a < 0$, we have a spiral sink. And when $a > 0$, we have a saddle point at the origin.

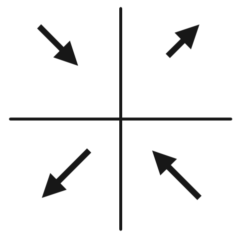
   b. So we have bifurcations at $a = -4$ (change from real to spiral sink) and $a = 0$ (change from spiral sink to saddle).

5. Here we have $T = a$ and $D = 4$, so our path is along the horizontal line $D = 4$. We cross the repeated eigenvalue parabola $T^2 - 4D = 0$ when $a^2 = 16$, so at $a = \pm 4$. When $a < -4$, we are in the real sink region. Then we cross into the spiral sink region. We cross the D-axis when $a = T = 0$ and then enter the spiral source region, and then finally enter the real source region when $a = 4$. So bifurcations occur when $a = \pm 4$ and $a = 0$.

Lecture 15

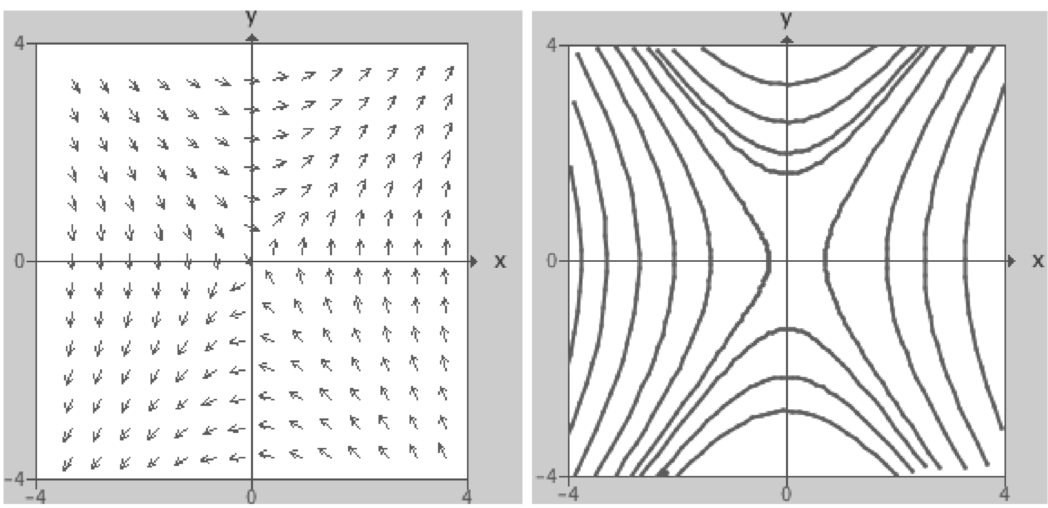
1. **a.** The x-nullcline is the x-axis, while the y-nullcline is the y-axis.

b.



c. The only equilibrium point is the origin.

d.



e. Since this system is linear and has real eigenvalues $\pm\sqrt{2}$, there is only one straight line through the origin containing solutions that tend to the origin.

2. The x-nullclines are given by the y-axis and the line $y = -x/3 + 50$. The y-nullclines are given by the x-axis and the line $y = -2x + 100$. Most solutions that start with x and y nonzero will tend to either the equilibrium point at (150, 0) or the one at (0, 100).

3. The x-nullcline is given by the y-axis and the line $y = 50 - x/2$. The y-nullclines are given by the x-axis and the line $y = 25 - x/6$. Most solutions will tend to the equilibrium point at (75, 12.5).

4. The x-nullclines are the lines $x = 0$ and $x = 1$, and the y-nullcline is the parabola $x = y^2$ opening to the right. So, there are equilibrium points at the origin, (1, 1), and (1, −1). We have $x' < 0$ when $x < 0$, so all solutions in the left half of the plane tend off to infinity. From the directions of the vector field, it appears that (1, 1) is a sink and (1, −1) is a saddle.

5. **a.** The x-nullclines are the y-axis and the line $y = 1/A - x/A$, and the y-nullclines are the x-axis and the line $y = 1 + x$. There are equilibrium points (1, 0), (0, 1), and (0, 0). When $A < 1$, solutions that begin with x and y nonzero tend to (1, 0). When $A > 1$, solutions tend to (0, 1). So a bifurcation occurs when $A = 1$. When $A = 1$, there is a straight line of equilibria along $y = 1 - x$.

   b. Now the x-nullclines are the y-axis and the line $y = 1 - x$, and the y-nullclines are the x-axis and $y = 1 + Bx$. Again the equilibrium points are at (1, 0), (0, 1), and (0, 0). Now if $B > -1$, all solutions tend to (0, 1) when the initial conditions are not on the x-axis, whereas if $B < -1$, these solutions tend to (1, 0) (unless the solution is on the y-axis).

Lecture 16

1. The partial derivative with respect to x is $2xy + 3x^2$ and with respect to y is $x^2 + 3y^2$.

2. The Jacobian matrix is just the coefficient matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

3. **a.** The only equilibrium point is at the origin.

b. The Jacobian matrix is

$$\begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}.$$

At the origin, this matrix becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

c. The eigenvalues of the Jacobian matrix are ± 1, so the origin is a saddle.

4. The only equilibrium point is at the origin, and when linearized, the system becomes

$$Y' = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} Y.$$

The characteristic equation is $\lambda^2 - \lambda + 1$. The eigenvalues are

$$\frac{1 \pm \sqrt{-3}}{2},$$

so the origin is a spiral source. Using a computer, you can see that all other solutions spiral toward a periodic solution that surrounds the origin.

5. One of many possibilities is $x' = x^2$, $y' = y^2$, which has a single equilibrium point at the origin. The Jacobian matrix is the zero matrix, so both eigenvalues are 0.

6. The equilibrium points are given by $y = 1$ and $x = \pm 1$. The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -2x & 1 \end{pmatrix}.$$

At (1, 1) the eigenvalues are $(1 \pm \sqrt{-7})/2$, so we have a spiral source. At (−1, 1) the eigenvalues are 2 and −1, so we have a saddle.

7. The equilibria are given by $(n\pi,\ \pi/2 + m\pi)$, where n and m are integers. Linearization shows that the equilibrium point is a saddle if n and m are both even (or both odd), a sink if n is odd and m is even, and a source if n is even and m is odd. The lines $x = n\pi$ are the x-nullclines, so the vector field is tangent to these lines, and solutions

remain on them. Similarly, the y-nullclines are the lines $y = \pi/2 + m\pi$, and the vector field is again tangent to these lines. These vertical and horizontal lines therefore bound squares whose corners are a pair of saddles, one sink and one source. In each square, all solutions not on the bounding lines tend to the equilibrium that is the sink.

8. The equilibrium points are at $y = 0$ and $x = \pm\sqrt{A}$, so we clearly have a bifurcation at $A = 0$. Linearization yields the eigenvalues 1 and $-2\sqrt{A}$ at the equilibrium point $(\sqrt{A}, 0)$, so this point is a saddle when $A > 0$. The eigenvalues at the other equilibrium point are 1 and $2\sqrt{A}$, so this point is a source when $A > 0$.

Lecture 17

1. **a.** The equilibria are (0, 0) and (1, 1).

b. The general Jacobian matrix is

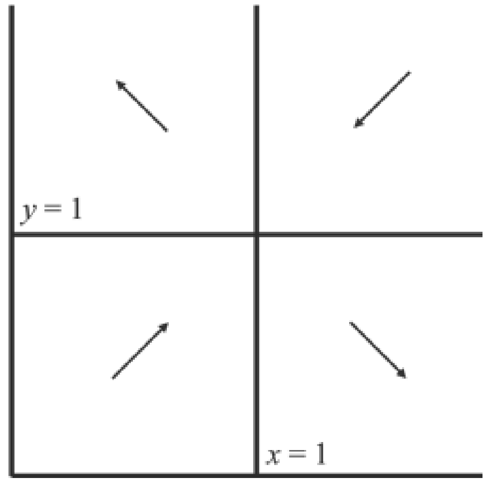
$$\begin{pmatrix} 1-y & -x \\ -y & 1-x \end{pmatrix}.$$

At (0, 0) and (1, 1) this matrix becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

c. At (0, 0) we have a source, and at (1, 1) we have a saddle.

d. The x-nullclines are $x = 0$ and $y = 1$. The y-nullclines are $y = 0$ and $x = 1$. The different regions are:



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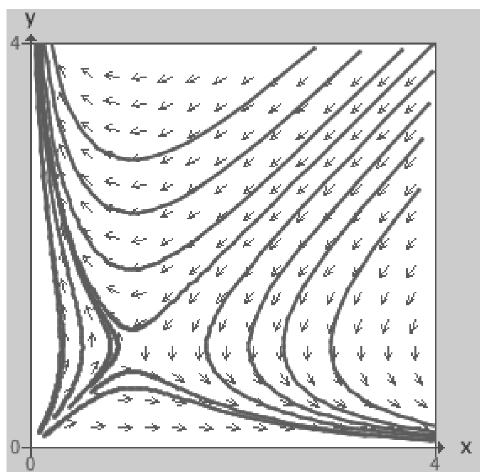


2. a. The coexistence equilibrium point when $a = b = 1/2$ is given by $x = y = 800/3$. The Jacobian matrix at this point is

$$\begin{pmatrix} -2/3 & -1/3 \\ -1/3 & -2/3 \end{pmatrix},$$

so the eigenvalues are the roots of $\lambda^2 + (4/3)\lambda + 1/3 = 0$. These eigenvalues are then -1 and $-1/3$, both of which are negative, so this equilibrium point is a sink.

b. The Jacobian matrix is given by

$$\begin{pmatrix} 1 - x/200 - ay/400 & -ax/400 \\ -by/400 & 1 - y/200 - bx/400 \end{pmatrix}.$$

At (0, 400), this matrix is

$$\begin{pmatrix} 1-a & 0 \\ -b & -1 \end{pmatrix},$$

so the eigenvalues are $1 - a$ and -1. Therefore, we have a sink if $a > 1$ and a saddle if $a < 1$. At the point (400, 0) we find eigenvalues -1 and $1 - b$, so this point is a sink if $b > 1$ and a saddle if $b < 1$.

3. **a.** The equilibria are (1, 0), and $x = y = 1/2$. (Technically, the equation for y' is not defined if $x = 0$).

b. The Jacobian matrix is

$$\begin{pmatrix} 1 - 2x - y & -x \\ y^2/x^2 & 1 - (2y/x) \end{pmatrix}.$$

At (1, 0) the eigenvalues are -1 and 1, so we have a saddle. And at the other equilibrium point, (1/2, 1/2), we have eigenvalues that are complex with negative real part, so this equilibrium is a spiral sink.

c. The x-nullclines are given by $x = 0$ and $1 - x = y$, and the y-nullclines are given by $y = 0$ and $y = x$. So the nullclines meet at a single point that is not on the axes. In the regions between the nullclines, we see that the vector field indicates that solutions spiral around the equilibrium point. But that does not tell us the complete story, as we could have periodic solutions in this region. But the computer shows otherwise.

Lecture 18

1. **a.** There are no limit cycles since for any point (except the origin), the vector field points directly northwest.

   b. All solutions lie along straight lines with slope equal to 1. Since the origin is the only equilibrium point, all solutions tend to infinity except for those on the line $y = x$, where x and y are less than 0. These solutions tend to the equilibrium point at the origin.

2. On the circle $x^2 + y^2 = 1$, the system reduces to

   $$x' = -y$$

   $$y' = x,$$

   which is a vector field that is everywhere tangent to the unit circle.

3. On the unit circle, the vector field is now given by

   $$x' = 0$$

   $$y' = x - y,$$

   so the vector field points vertically on this circle, which means that the unit circle is not a periodic solution.

4. The second equation says that $x = xy/(1 + x^2)$. Substituting this into the first equation yields $x - 10 = -4x$, or $x = 2$. Then the second equation gives $2 = 2y/5$ so that $y = 5$.

5. **a.** We have $r' = 0$ when $r = 1$, so there is a periodic solution along the unit circle. (At the origin, we have an equilibrium point.) Since $\theta' = 1$, this solution moves counterclockwise around the circle. When $0 < r < 1$, we have $r' > 0$, and when $r > 1$, $r' < 0$, so all nonzero solutions spiral in to this limit cycle at $r = 1$, which is therefore stable.

b. We now have periodic solutions when $r^3 - 3r^2 + 2r = 0$ and when $r(r - 2)(r - 1) = 0$. That is, $r = 1$ and 2 are periodic solutions as in the previous question. Choosing one point in each of the intervals between these circles, we see that $r' > 0$ when $0 < r < 1$; $r' < 0$ when $1 < r < 2$; and $r' > 0$ when $r > 2$. So $r = 1$ is a stable limit cycle while $r = 2$ is unstable.

c. In similar fashion, we have a limit cycle at $r = n\pi$, where n is a positive integer. Odd integers yield stable limit cycles; even integers yield unstable limit cycles. The rotation is now in the clockwise direction.

6. The roots of $ar - r^2 + r^3 = r(r^2 - r + a)$ are 0 and

$$r_{\pm} = \frac{1 \pm \sqrt{1-4a}}{2}.$$

Besides the equilibrium point at the origin, there are two limit cycles when $a < 1/4$, a single limit cycle when $a = 1/4$, and no limit cycles when $a > 1/4$. The limit cycle $r = r_-$ is stable; $r = r_+$ is unstable. When $a > 1/4$, $r' > 0$, so all nonzero solutions spiral out to infinity.

Lecture 19

1. When the pendulum is in the downward position (θ = 0).

2. When the pendulum is in the upward position (θ = π).

3. The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -g\cos(\theta) & 0 \end{pmatrix}, \text{ so when } \theta = \pi, \text{ we have the matrix } \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix}.$$

The eigenvalues for this matrix are $\pm\sqrt{g}$, so this equilibrium point is a saddle.

4. Now the eigenvalues are $\pm i\sqrt{g}$, so linearization does not give any information. But we know that the system is Hamiltonian, and the level curves surrounding this equilibrium point are ellipses, so this equilibrium point is a center.

5. Since $\theta = 2n\pi$, the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -g & -b \end{pmatrix}$$

with characteristic equation $\lambda^2 + b\lambda + g = 0$ and roots

$$\frac{-b \pm \sqrt{b^2 - 4g}}{2}.$$

If $b^2 > 4g$, the roots are both real and negative, so we have a real sink. If $0 < b^2 < 4g$, the roots are complex with negative real part, so we have a spiral sink.

6. The Jacobian matrix now has a $+g$ in the lower left entry, so the determinant is now $-g < 0$. This means the equilibrium point is a saddle.

7. We have $\partial F/\partial x = 2x = -\partial G/\partial y$, so this system is Hamiltonian.

8. Let $F(x, y) = ax + by$ and $G(x, y) = cx + dy$. This linear system is Hamiltonian if $\partial F/\partial x = a = -d = -\partial G/\partial y$. The trace of the corresponding matrix is then equal to 0, so we can only have a center or a saddle equilibrium point (or repeated 0 eigenvalues).

9. The Hamiltonian function is given by $H(x, y) = by^2/2 + axy - cx^2/2$ (plus possibly a constant).

Lecture 20

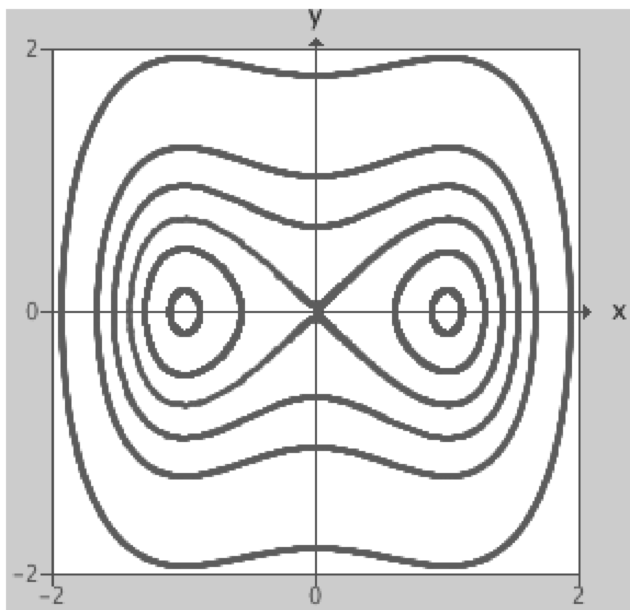
1. **a.** (0, 0), (1, 0), and (−1, 0).

   b. Computing the respective partial derivatives

   $$\frac{\partial H}{\partial v} = v, -\frac{\partial H}{\partial y} = y - y^3$$

   shows that the system is Hamiltonian.

   c.



   d. If the beam starts to the right or left of the center of the two magnets with small velocity, then it just oscillates back and forth in either the left or right region, depending on its starting position. But if the beam starts with large velocity, then it will move periodically back and forth to the left and right regions.

2. The function $x(t) = 0$, $y(t) = 0$, and $z(t) = Ce^{-8/3t}$ is such a solution.

3. We first compute that $dL/dt = -20(x^2 + y^2 - (1 + R)xy) - (160/3)z^2$. So we need to show that the term $x^2 + y^2 - (1 + \mathrm{R})xy$ is always positive away from the origin. This is certainly true when $x = 0$. Along any other straight line $y = Mx$, this quantity is equal to $x^2(M^2 - (1 + R)M + 1)$, but the quadratic term $M^2 - (1 + R)M + 1$ is always positive if $R < 1$.

4. All solutions must tend to $(0, 0, 0)$ when $R < 1$.

5. We compute $dV/dt = -20(Rx^2 + y^2 + (8/3)(z^2 - 2Rz)) = -20(Rx^2 + y^2 + (8/3)(z - R)^2) = (8/3)R^2$. Note that the equation $Rx^2 + y^2 + (8/3)(z - R)^2 = K$ defines an ellipsoid when $K > 0$. So if $K > (8/3)R^2$, we have $dV/dt < 0$. So we may choose a constant C large enough so that the ellipsoid $V = C$ strictly contains the ellipsoid $Rx^2 + y^2 + (8/3)(z - R)^2 = (8/3)R^2$ in its interior. Then we have $dV/dt < 0$ on the ellipsoid $V = C + \alpha$ for any constant $\alpha > 0$.

6. Far enough away on the ellipsoid $V = C + \alpha$, for example, all solutions descend toward the ellipsoid $V = C$.

Lecture 21

1. **a.** 0 and 1.

   b. If $|x| < 1$, the orbit of x tends to the fixed point at 0. If $|x| > 1$, the orbit tends to infinity. If $x = -1$, the orbit lands on 1 after 1 iteration and so is eventually fixed.

2. All orbits tend to infinity.

3. **a.** All orbits tend to the fixed point at 0.

   b. For each a, 0 is always fixed. So we consider orbits of other x-values. If $a > 1$, all these orbits tend to infinity. If $a = 1$, all these orbits are fixed. If $a = -1$, all these orbits lie on 2-cycles. If $-1 < a < 0$, all these orbits tend to 0. And if $a < -1$, all orbits tend to $\pm\infty$, alternating between the positive and the negative axis.

4. The fixed points are 0 and $(k - 1)/k$ (which only exists in the unit interval if $k > 1$).

5. The points 1 and -1 lie on a 2-cycle for $-x^3$. There is no 2-cycle for x^3 since the graph of x^3 is always increasing.

6. $1/3 \to 2/3 \to 1/3$, so $1/3$ lies on a 2-cycle. $1/7 \to 2/7 \to 4/7 \to 1/7$, so $1/7$ lies on a 3-cycle. $1/15 \to 2/15 \to 4/15 \to 8/15 \to 1/15$, so $1/15$ lies on a 4-cycle.

7. Only 1/3 and 2/3 have prime period 2. The points 1/7 … 6/7 have prime period 3, and 1/15, … , 14/15 have prime period 4 (with the exception of 5/15 and 10/15, which have prime period 2).

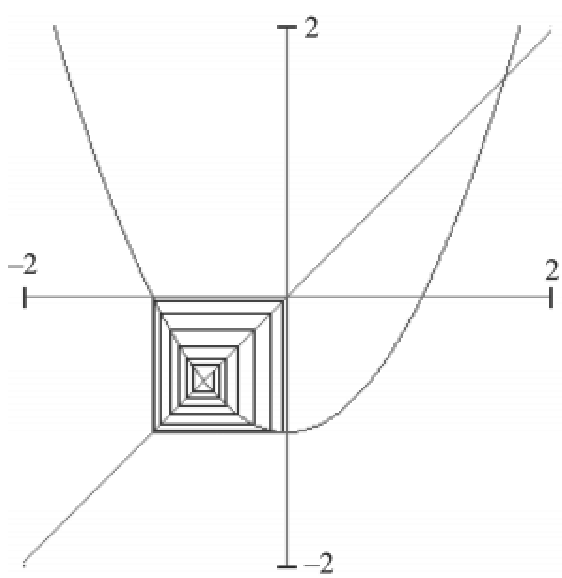
8. The graph of D crosses the diagonal two times, D^2 four times, D^3 eight times, and D^n 2^n times. Points of the form $k/(2^n - 1)$, where k is an integer and $1 \leq k < 2^n - 1$, have (not necessarily prime) period n.

Lecture 22

1. The fixed points are 0 (attracting) and ±1 (repelling).

2. **a.** This cycle is attracting.

b.



3. **a.** If $-1 < a < 1$, 0 is attracting; if $a > 1$ or $a < -1$, 0 is repelling. If $a = 1$, all other orbits are fixed, or if $a = -1$, all other orbits lie on two-cycles. So, in both of these cases, 0 is neutral.

   b. Bifurcations occur at $a = 1$ and $a = -1$.

4. The derivative of the logistic function is $k - 2kx$. So the fixed point at 0 is attracting for $0 < k \leq 1$ and repelling for $k > 1$. The other fixed point is attracting if $1 < k \leq 3$ and repelling if $k > 3$.

5. If $F(x) = -x^3$, then $F^2(x) = x^9$, and we know 1 and -1 lie on a 2-cycle. But the derivative of $F^2(x)$ is $9x^8$, so this cycle is repelling.

6. We have $D' = 2$ at all points, so the derivative of D^n is 2^n everywhere, so all cycles are repelling.

7. The derivative of the logistic function at the fixed point $(k - 1)/k$ is equal to -1 when $k = 3$, so this should be the place where the bifurcation occurs. The graph of F^2 then shows the emergence of two new fixed points as k increases through 3.

8. This function always has a fixed point at $x = 0$, and the derivative at this point is equal to a. When $a > 1$, all other points have orbits that tend to infinity. When $a < 1$, there are two more fixed points at $\pm\sqrt{1-a}$. When $a = -1$, the derivative at 0 becomes -1. For $a < -1$, there is a 2-cycle at a pair of symmetrically located points given by $\pm\sqrt{-(a+1)}$.

Lecture 23

1. **a.** (1/27, 2/27), (7/27, 8/27), (19/27, 20/27), and (25/27, 26/27).

   b. The removed lengths are 1/3, 2/9, and 4/27, which add up to 19/27.

   c. At stage 4, the removed intervals have total length $8/81 = 2^3/3^4$.

   d. At stage n, the removed intervals have length $2^{n-1}/3^n$.

   e. The infinite series is

$$\frac{1}{3}+\frac{2}{9}+\ldots+\frac{2^{n-1}}{2^n}+\ldots=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n-1}=\frac{1}{3}\frac{1}{1-\frac{2}{3}}=1,$$

   so the length of the remaining Cantor set is 0.

2. Any sequence that ends in all zeros eventually lands on (0, 0, 0, …) under the shift map, so any sequence of the form (s_0, s_1, s_2, … , s_n 0000 …) has this property.

3. The sequences above correspond to the points that lie at the endpoints of the intervals we called A_n.

4. If you just interchange all the blocks of length n in the original sequence, you get a new point in the sequence space whose orbit is dense. Or you could just take the original sequence and throw in a 0 before each block. There are countless different ways to find such interesting orbits.

5. For example, (0100100010000100000 1…).

6. There are 3^n points that are fixed under the n^{th} iterate of the shift map in this case; they correspond to all possible blocks of length n, which are then repeated infinitely often in the sequence space to get the periodic point. As in the case of 2 symbols, if we use the sequence that consists of all possible blocks of length 1 in order (i.e., 0, 1, and 2), then all possible blocks of length 2 (00, 01, 02, 10, 11, etc.) have a dense orbit.

Lecture 24

1. $i \to -1 \to 1 \to 1$ …, so this orbit is eventually fixed.

2. $2i \to -4 \to 16 \to 256 \to$ …, so this orbit tends to infinity.

3. $i/2 \to -1/4 \to 1/256 \to$ …, so this orbits goes to zero.

4. Any point $z = x + iy$ where $x^2 + y^2 > 1$ (i.e., z lies outside the unit circle) has an orbit that tends to infinity.

5. Any point z inside the unit circle has an orbit that tends to the fixed point at the origin.

6. The fixed points always exist in the complex domain and are given by the roots of $z^2 - z + c = 0$, or

$$\frac{1 \pm \sqrt{1-4c}}{2}.$$

Note that these fixed points are complex when $c > 1/4$.

7. The orbit of 0 eventually lands on the 2-cycle given by $-1 + i$ and $-i$, so the filled Julia set is connected.

8. The filled Julia sets are always a scatter of points when A is non-zero. When $A = 0$, this is the only place where the filled Julia set is connected. When $|A| \neq 0$ is small, it may appear that the filled Julia set consists of a single piece, but changing the number of iterations to be a much larger number shows that this is not the case.

9. Each window in the orbit diagram corresponds to a baby Mandelbrot set.

10. As in question 6, the fixed points still exist when c moves above 1/4. But now the filled Julia set immediately shatters into a totally disconnected set as soon as c increases above 1/4.

Types of Differential Equations Cited

FIRST ORDER

Linear: can be homogenous or nonhomogeneous, autonomous or nonautonomous.

	Autonomous	Nonautonomous
Homogeneous	$y' + ky = 0$	$y' + G(t)y = 0$
Nonhomogeneous	$y' + ky = 2$	$y' + G(t)y = 2t$

Nonlinear:

Autonomous: $y' = y(1 - y)$

Nonautonomous: $y' = t^2 + y^2$

SECOND ORDER

Linear: can be homogenous or non-homogeneous, autonomous or nonautonomous.

	Autonomous	Nonautonomous
Homogeneous	$y'' + by' + ky = 0$	$y' + G(t)y = 0$
Nonhomogeneous	$y'' + by' + ky = 2$	$y' + G(t)y = 2$

Nonlinear:

Autonomous: $y'' = y^2$

Nonautonomous: $y'' = t^2$

SYSTEMS

Linear: $Y' = AY = -x' = ax + by$

$y' = cx + dy$

Nonlinear: $x' = y$

$y' = \sin(x)$

	Autonomous	Nonautonomous
Linear	$Y' = AY$ where A is a constant matrix	$Y' = AY + (\sin(t), \cos(t))$
Nonlinear	$Y' = (x^2, \sin(y))$	$Y' = F(Y) = (x^2 + t, \sin(y) + \cos(t))$

Both linear and nonlinear systems can be autonomous or nonautonomous; however, we did not deal with the nonautonomous cases in this course.

Using a Spreadsheet to Solve Differential Equations

Here are some tips for creating spreadsheets as introduced in Lecture 6: "How Computers Solve Differential Equations." As of this writing, most steps were virtually the same for both Mac and PC users, and for Microsoft Excel and OpenOffice; however, do consult your software documentation if you have difficulties.

To enter a formula in a cell:

First type "=" and then enter the corresponding formula, clicking on the cell row/column to indicate the variables. Use * for multiplication.

	A	B	C
1	10	5	3
2	**=A1*C1**		
3			

For a constant reference to a cell, say cell B1, type "B1."

	A	B	C
1	10	5	3
2			
3	**=B1**		

After hitting Enter:

	A	B	C
1	10	5	3
2	**30**		
3	**5**		

This works on a PC with Excel 2007 as well as the spreadsheet program available in OpenOffice.

To fill down a formula:

Whether in Excel or OpenOffice, grab the lower right corner of the cell or cells you want to fill down, then drag it down as far as you wish. (Be careful not to <u>move</u> the data from the first cell into the next cell.) In this process, cell references will automatically increase, e.g., G3 will change to G4 in the next row, to G5 in the second row, and so forth. But constant references such as "F3" will not change; that is, just insert a "$" before the letter, and another "$" before the number, of any cell that you want to remain unchanged across calculations to determine more than one cell.

To insert a chart:

As demonstrated in Lecture 6, first highlight the data you wish to plot. In that lecture, I wanted to plot the *t* and *y* values generated by Euler's method, so I highlighted all the entries containing these two values. Then click on Insert and choose Chart. Many chart options will be shown. I chose the scatter plot for the Euler's method spreadsheet. In that case, we then had a choice of how to connect the dots; select the appropriate type of plot. (Also, though not demonstrated in the lecture, you can then modify the size and

color of the lines as well as the scale of the axes by clicking on the object in the plot that you wish to modify.)

Another way to do this in Excel 2007 for PC is to first insert a blank chart and then select the data you wish to use. To insert a blank chart, go to the Insert tab and select the type of Chart you want. Once you've inserted the chart, you will be given Design options in the Chart Tools toolbar, and you can select the option to Select Data.

To insert a scrollbar:

This can be a little complicated, but many people who have seen the demonstration become curious about how to do this. In Excel, select View → Toolbars → Forms. For OpenOffice, select View → Toolbars → Form Controls. Then choose the scrollbar from this menu. To insert the scrollbar into your spreadsheet, highlight the area where you wish to place the scrollbar. This could be a horizontal or vertical rectangle.

For Excel 2007 for PC, you must in addition have the Developer tab displayed. To display this tab, click on the circular Microsoft Office button in the top left corner of the spreadsheet, which opens a list of options and your recent documents, and then click on the Excel Options button at the bottom of the window. Once you are in the Excel Options window, select Popular from the menu on the left, and check the box for "Show Developer tab in the Ribbon." It should be listed among the "Top options for working with Excel." Click OK.

Once the Developer tab is displayed in Excel, go to it and click on Insert and then select the scrollbar option under Form Controls. To find the scrollbar, hover your mouse over the various options and read the descriptions that pop up for each icon. Click on the scrollbar icon and then click in the spreadsheet where you would like the scrollbar to appear.

To resize the scrollbar:

- For Excel, click on the scrollbar to reveal the points around the outline. Place your mouse over the point in the bottom right corner of the scrollbar so that an arrow appears. Click on that point and drag the edge of the scrollbar in or out until it is the appropriate size. The default orientation is vertical, but you can change it to a horizontal orientation. To change the orientation from vertical to horizontal, follow the same steps as for resizing but drag the bottom right edge of the scrollbar up and to the right at the same time until its orientation shifts.

- For OpenOffice, first make sure the Design Mode On/Off button in the Form Controls toolbar is in the On mode (the button will appear highlighted), and then click on the scrollbar button. Then, using your cursor, highlight the cells where you wish to place the scrollbar, and it should appear. Following the same steps as above for Excel, you can drag it by the corners to resize it.

To arrange for the appropriate output of the scrollbar:

For example, in our Euler's method spreadsheet, I wanted the output to be the value of delta *t*, which moved from .01 to 1.01 in steps of size .01. That is, I wanted to have 100 different values of delta *t* as I manipulated the scrollbar. To accomplish this, you need to "format" the scrollbar. To do this, while the scrollbar you inserted is highlighted, select Format → Control.

- For Excel, right-click on the scrollbar and select Format Control.

- For OpenOffice, right click on the scrollbar and select Control, which opens the Scrollbar Properties window. Again, the Design Mode On/Off button in the Form Controls toolbar must be in the On mode (highlighted).

One immediate problem is that a scrollbar only puts out nonnegative integers. So you will need to use a little algebra to change the types of outputs.

- The first thing to do is to select a cell link where the output will be placed. Since the output is an integer and we want something else, I would choose an output cell link that is off the given spreadsheet page, such as cell Z25. To assign the output cell, enter that cell (say it's Z25) in the Cell Link field of the Format Control window.

- For OpenOffice, select the Data tab in the Scrollbar Properties window and enter Z25 in the Linked Cell field.

To choose a minimum and maximum output for the spreadsheet:

- In our case, we wanted the steps to go from .01 to 1.01 in 100 different steps. So our minimum value would be 0 and the maximum would be 100. If this is not the default in the Minimum value and Maximum value fields, enter "0" and "100," respectively. Once you have entered these values, click OK, and when you move the scrollbar, you will see the entries in cell Z25 move from 0 to 100.

- For OpenOffice, enter these values in the Scroll value min and Scroll value max fields in the Data tab of the Scrollbar Properties window. For OpenOffice, there is no OK button; simply close the Scrollbar Properties window. NOTE: To test the scrollbar functionality, make sure the Design Mode On/Off button in the Form Controls toolbar is in the Off mode.

Since we want the entries to change from .01 to 1.01, we then enter in some cell that's visible, say cell F2, the formula =.01*Z25 + .01 and hit Enter.

	A	B	C	D	E	F	G
1							
2						=0.01*Z25 + 0.01	
3							

After hitting Enter:

	A	B	C	D	E	F	G
1							
2						0.01	
3							

Then, as you click down the scrollbar (whether in Excel or OpenOffice), you see the entries in cell F2 change from .01 to 1.01 as desired.

After clicking down once on the scrollbar:

	A	B	C	D	E	F	G
1							
2						0.02	
3							

After dragging the scrollbar all the way to the bottom:

	A	B	C	D	E	F	G
1							
2						1.01	
3							

Timeline

1693 .... Isaac Newton's "fluxions" and Gottfried Leibniz's "calculus of differences" offer the first published solutions to differential equations.

1700s.. Followers of Leibniz—including Daniel Bernoulli, Joseph Lagrange, Pierre Laplace, and Leonhard Euler—continue the development of "differential calculus," while Newton's "calculus of fluxions" remains more influential in England.

1768 .... Euler develops a method for approximating the solution to a differential equation.

1835 .... William Rowan Hamilton defines the special system of differential equations in which there is a function that is constant along all solutions of the given differential equation.

1841 .... Carl Jacobi advances the study of determinants.

1858 .... Arthur Cayley initiates the use of matrices to reduce systems of differential equations to

1866 .... Jacobi refines Hamiltonian functions and pioneers the Jacobian matrix for evaluating systems of partial derivatives.

1880s... A geometric approach to nonlinear differential equations by Henri Poincaré initiates the qualitative theory of differential equations.

1890s... Aleksandr Lyapunov defines a function that is nonincreasing along all solutions of a system of differential equations.

1901 .... Poincaré's conjecture for showing the existence of a limit cycle is proven by Ivar Bendixson.

1895–1905 .... Alternative methods for iterative and numerical solutions of differential equations are offered by Carl Runge and Martin Kutta.

1918 .... Gaston Julia and Pierre Fatou pioneer iteration theory and lay a foundation for work on fractals.

1931 .... The differential analyzer, an electrically-powered mechanical device for solving differential equations, is invented by Vannevar Bush and others at M.I.T.

1942	An M.I.T. differential analyzer with 2000 vacuum tubes begins to make important contributions during WWII.
1950	The first digital difference analyzer is created by Northrop Corporation, a leading U.S. aircraft manufacturer.
1950s	Boris Belousov discovers that certain chemical reactions could oscillate rather than go to equilibrium.
1963	While studying an extremely simplified problem in meteorology, Edward Lorenz offers the first system of differential equations shown to possess chaotic behavior.
1960s	Russian mathematician Anatol Zhabotinsky revives Belousov's work on oscillating chemical reactions, with their findings beginning to diffuse to other countries in 1968.
1960s	Sharkovsky's Theorem, describing the ordering of cycle periods for any iterated function, is published in Russian.
1975	A paper by Tien-Yien Li and James Yorke proves that the presence of a periodic point of period 3 implies the presence of periodic points of all other periods, and therefore chaos.

1976 .... An influential paper published in *Nature* by biologist Robert May encourages mathematicians to turn from an exclusive emphasis on higher-dimensional differential equations and give more attention to the complicated dynamics in low-dimensional difference equations such as the logistic difference equation.

1970s.. Benoit Mandelbrot's work on fractals and use of computers extends the work of Julia and Fatou on iterated functions, drawing worldwide attention to visual solutions of real-number problems using the complex plane.

1980s.. Eminent mathematicians such as Dennis Sullivan, John Milnor, and Curtis McMullen make major advances in the field of complex dynamics, motivated by Mandelbrot's introduction of the set that bears his name.

1998 .... Swedish mathematician Warwick Tucker proves the existence of chaotic attractor in the Lorenz equations.

Glossary

beating modes: The type of solutions of periodically forced and undamped mass-spring systems that periodically have small oscillations followed by large oscillations.

bifurcation: A major change in the behavior of a solution of a differential equation caused by a small change in the equation itself or in the parameters that control the equation. Just tweaking the system a little bit causes a major change in what occurs. *See also* **Hopf bifurcation**, **pitchfork bifurcation**, and **saddle-node bifurcation**.

bifurcation diagram (bifurcation plane): A picture that contains all possible phase lines for a first-order differential equation, one for each possible value of the parameter on which the differential equation depends. The bifurcation diagram, which plots a changing parameter horizontally and the y value vertically, is similar to a parameter plane, except that a bifurcation diagram includes the dynamical behavior (the phase lines), while a parameter plane does not.

carrying capacity: In the limited population growth population model, this is the population for which any larger population will necessarily decrease, while any smaller population will necessarily increase. It is the ideal population.

center: An equilibrium point for a system of differential equations for which all nearby solutions are periodic.

characteristic equation: A polynomial equation (linear, quadratic, cubic, etc.) whose roots specify the kinds of exponential solutions (all of the form e^{At}) that arise for a given linear differential equation. For a second-order linear differential equation (which can be written as a 2-dimensional linear system of differential equations), the characteristic equation is a quadratic

equation of the form $\lambda^2 - T\lambda + D$, where T is the trace of the matrix and D is the determinant of that matrix.

critically damped mass-spring: A mass-spring system for which the damping force is just at the point where the mass returns to its rest position without oscillation. However, any less of a damping force will allow the mass to oscillate.

damping constant: A parameter that measures the air resistance (or fluid resistance) that affects the behavior of the mass-spring system. Contrasts with the **spring constant**.

determinant: The determinant of a 2-by-2 matrix is given by $ad - bc$, where a and d are the terms on the diagonal of the matrix, while b and c are the off-diagonal terms. The determinant of a matrix A (det$A$, for short) tells us when the product of matrix A with vector Y to give zero ($A\ Y = 0$) has only one solution (namely the 0 vector, which occurs when the determinant is non-zero) or infinitely many solutions (which occurs when the determinant equals 0).

difference equation: An equation of the form $y_{n+1} = F(y_n)$. That is, given the value y_n, we determine the next value y_{n+1} by simply plugging y_n into the function F. Thus, the successive values y_n are determined by iterating the expression $F(y)$.

direction field: This is the vector field each of whose vectors is scaled down to be a given (small) length. We use the direction field instead of the vector field because the vectors in the vector field are often large and overlap each other, making the corresponding solutions difficult to visualize. Those solutions and the direction field appear within the phase plane. *See also* **vector field**.

eigenvalue: A real or complex number usually represented by λ (lambda) for which a vector Y times matrix A yields non-zero vector λY. In general, an $n \times n$ matrix will have n eigenvalues. Such values (which have also been called proper values and characteristic values) are roots of the

corresponding characteristic equation. In the special case of a triangular matrix, the eigenvalues can be read directly from the diagonal, but for other matrices, eigenvalues are computed by subtracting λ from values on the diagonal, setting the determinant of that resulting matrix equal to zero, and solving that equation.

eigenvector: Given a matrix A, an eigenvector is a non-zero vector that, when multiplied by A, yields a single number λ (lambda) times that vector: $AY = \lambda Y$. The number λ is the corresponding eigenvalue. So, when λ is real, AY scales the vector Y by a factor of lambda so that AY stretches or contracts vector Y (or $-Y$ if lambda is negative) without departing from the line that contains vector Y. But λ may also be complex, and in that case, the eigenvectors may also be complex vectors.

equilibrium solution: A constant solution of a differential equation.

Euler's formula: This incredible formula provides an interesting connection between exponential and trigonometric functions, namely: The exponential of the imaginary number $(i \cdot t)$ is just the sum of the trigonometric functions $\cos(t)$ and $i \sin(t)$. So $e^{(it)} = \cos(t) + i \sin(t)$.

Euler's method: This is a recursive procedure to generate an approximation of a solution of a differential equation. In the first-order case, basically this method involves stepping along small pieces of the slope field to generate a "piecewise linear" approximation to the actual solution.

existence and uniqueness theorem: This theorem says that, if the right-hand side of the differential equation is nice (basically, it is continuously differentiable in all variables), then we know we have a solution to that equation that passes through any given initial value, and, more importantly, that solution is unique. We cannot have two solutions that pass through the same point. This theorem also holds for systems of differential equations whenever the right side for all of those equations is nice.

filled Julia set: The set of all possible seeds whose orbits do not go to infinity under iteration of a complex function.

first derivative test for equilibrium points: This test uses calculus to determine whether a given equilibrium point is a sink, source, or node. Basically, the sign of the derivative of the right-hand side of the differential equation makes this specification; if it is positive, we have a source; negative, a sink; and zero, we get no information.

fixed point: A value of y_0 for which $F(y_0) = y_0$. Such points are attracting if nearby points have orbits that tend to y_0; repelling if the orbits tend away from y_0; and neutral or indifferent if y_0 is neither attracting or repelling.

general solution: A collection of solutions of a differential equation from which one can then generate a solution to any given initial condition.

Hopf bifurcation: A kind of bifurcation for which an equilibrium changes from a sink to a source (or vice versa) and, meanwhile, a periodic solution is born.

initial value problem: A differential equation with a collection of special values for the missing function such as its initial position or initial velocity.

itinerary: An infinite string consisting of digits 0 or 1 that tells how a particular orbit journeys through a pair of intervals I_0 and I_1 under iteration of a function.

Jacobian matrix: A matrix made up of the various partial derivatives of the right-hand side of a system evaluated at an equilibrium point. The corresponding linear system given by this matrix often has solutions that resemble the solutions of the nonlinear system, at least close to the equilibrium point.

limit cycle: A periodic solution of a nonlinear system of differential equations for which no nearby solutions are also periodic. Compared with linear systems, where a system that has one periodic solution will always have infinitely many additional periodic solutions, the limit cycle of a nonlinear system is isolated.

limited population growth model with harvesting: This is the same as the limited population growth model except we now assume that a portion of the population is being harvested. This rate of harvesting can be either constant or periodic in time.

Lyapunov function: A function that is non-increasing along all solutions of a system of differential equations. Therefore, the corresponding solution must move downward through the level sets of the function (i.e., the sets where the function is constant). Such a function can be used to derive the stability properties at an equilibrium without solving the underlying equation.

mass-spring system: Hang a spring on a nail and attach a mass to it. Push or pull the spring and let it go. The mass-spring system is a differential equation whose solution specifies the motion of the mass as time goes on. The differential equation depends on two parameters, the spring constant and the damping constant. A mass-spring system is also called a harmonic oscillator.

Newton's law of cooling: This is a first-order ODE that specifies how a heated object cools down over time in an environment where the ambient temperature is constant.

node: An equilibrium solution of a first-order differential equation that has the property that it is neither a sink nor a source.

orbit: In the setting of a difference equation, an orbit is the sequence of points x_0, x_1, x_2, ... that arise by iteration of a function F starting at the seed value x_0.

orbit diagram: A picture of the fate of orbits of the critical point for each value of a given parameter.

ordinary differential equation (ODE): A differential equation that depends on the derivatives of the missing functions. If the equation depends on the partial derivatives of the missing functions, then that is a partial differential equation (PDE).

phase line: A pictorial representation of a particle moving along a line that represents the motion of a solution of an autonomous first-order differential equation as it varies in time. Like the slope field, the phase line shows whether equilibrium solutions are sinks or sources, but it does so in a simpler way that lacks information about *how quickly* solutions are increasing or decreasing. The phase line is primarily a teaching tool to prepare students to make use of phase planes and higher-dimensional phase spaces.

phase plane: A picture in the plane of a collection of solutions of a system of two first-order differential equations: $x' = F(x, y)$ and $y' = G(x, y)$. Here each solution is a parametrized curve, $(x(t), y(t))$ or $(y(t), v(t))$. The term "phase plane" is a holdover from earlier times when the state variables were referred to as phase variables.

pitchfork bifurcation: In this bifurcation, varying a parameter causes a single equilibrium to give birth to two additional equilibrium points, while the equilibrium point itself changes from a source to a sink or from a sink to a source.

predator-prey system: This is a pair of differential equations that models the population growth and decline of a pair of species, one of whom is the predator, whose population only survives if the population of the other species, the prey, is sufficiently large.

resonance: The kind of solutions of periodically forced and undamped mass-spring systems that have larger and larger amplitudes as time goes on. This occurs when the natural frequency of the system is the same as the forcing frequency.

saddle-node bifurcation: In an ODE, this is a bifurcation at which a single equilibrium point suddenly appears and then immediately breaks into two separate equilibria. In a difference equation, a fixed or periodic point undergoes the same change. A saddle-node bifurcation is also referred to as a tangent bifurcation.

saddle point: An equilibrium point that features one curve of solutions moving away from it and one other curve of solutions tending toward it.

separation of variables: This is a method for finding explicit solutions of certain first-order differential equations, namely those for which the dependent variables (y) and the independent variables (t) may be separated from each other on different sides of the equation.

separatrix: The kind of solution that begins at a saddle point of a planar system of differential equations and tends to another such point as time goes on.

shift map: The map on a sequence space that just deletes the first digit of a given sequence.

sink: An equilibrium solution of a differential equation that has the property that all nearby solutions tend toward this solution.

solution curve (or graph): A graphical representation of a solution to the differential equation. This could be a graph of a function $y(t)$ or a parametrized curve in the plane of the form $(x(t), y(t))$.

source: An equilibrium solution of a differential equation that has the property that all nearby solutions tend away from this solution.

spiral sink: An equilibrium solution of a system of differential equations for which all nearby solutions spiral in toward it.

spiral source: An equilibrium solution of a system of differential equations for which all nearby solutions spiral away from it.

spring constant: A parameter in the mass-spring system that measures how strongly the spring pulls the mass. Contrasts with the **damping constant**.

steady-state solution: A periodic solution to which all solutions of a periodically forced and damped mass-spring system tend.

Taylor series: Method of expanding a function into an infinite series of increasingly higher-order derivatives of that function. This is used, for example, when we approximate a differential equation via the technique of linearization.

trace: The sum of the diagonal terms of a matrix from upper left to lower right.

vector field: A collection of vectors in the plane (or higher dimensions) given by the right-hand side of the system of differential equations. Any solution curve for the system has tangent vectors that are given by the vector field. These tangent vectors (and, even more so, the scaled-down vectors of a corresponding direction field) are the higher-dimensional analogue of slope lines in a slope field. *See also* **direction field**.

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